

DEFINITION OF CYLINDRICAL CONTACT HOMOLOGY IN DIMENSION THREE

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ABSTRACT. In this paper we give a rigorous definition of cylindrical contact homology for contact 3-manifolds that admit nondegenerate contact forms with no contractible Reeb orbits. By “defining contact homology” we mean the following: To a contact 3-manifold (M, ξ) we assign an isomorphism class $\{HC(\mathcal{D})\}_{\mathcal{D}}$ of groups, where each group $HC(\mathcal{D})$ is defined using some auxiliary data \mathcal{D} and any two groups $HC(\mathcal{D}^1)$ and $HC(\mathcal{D}^2)$ for the same (M, ξ) are naturally isomorphic.

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1. INTRODUCTION

The goal of this paper is to give a rigorous definition of cylindrical contact homology for contact 3-manifolds that admit nondegenerate contact forms with no contractible Reeb orbits. By “defining cylindrical contact homology” we mean the following:

Theorem 1.0.1. *If (M, ξ) is a closed oriented contact 3-manifold that admits nondegenerate contact forms with no contractible Reeb orbits, then there exists an isomorphism class $HC(M, \xi) = \{HC(\mathcal{D})\}_{\mathcal{D}}$ of groups, where each group $HC(\mathcal{D})$*

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is defined using auxiliary data

$$\mathcal{D} = (\alpha, \{L_i\}, \{\varphi_i\}, \{J_i\}, \{\bar{J}_i\})$$

and any two groups $HC(\mathcal{D}^1)$ and $HC(\mathcal{D}^2)$ for the same (M, ξ) are naturally isomorphic. Here α is a nondegenerate contact form for ξ with no contractible Reeb orbits and the rest of \mathcal{D} is described in Section 5.3.

The notion of contact homology was proposed by Eliashberg-Givental-Hofer in [EGH] over a decade ago, but a rigorous definition had not been written down yet, even for cylindrical contact homology for contact 3-manifolds. This is starting to change with our work and also the recent work of Hutchings-Nelson [HN] towards defining cylindrical contact homology in dimension three.

There are earlier “abstract perturbation” approaches which are much broader in scope and are likely to give a definition of contact homology: polyfolds of Hofer-Wysocki-Zehnder [HWZ3], Kuranishi structures of Fukaya-Ono [FO], and work of Liu-Tian [LT] and Ruan [Ru]. Our approach is quite different (and closer in spirit to [HN]) in that we do not use any type of abstract perturbation theory: we try to minimize the analysis by more carefully using asymptotic eigenfunctions in the spirit of Hutchings-Taubes [HT1, HT2].

Remark 1.0.2. The assumption of the theorem is designed to simplify the possible degenerations that we need to analyze. Taking away the assumption would lead to the next level of difficulty: defining full contact homology in dimension three.

Outline of proof. Starting with a nondegenerate contact form α with no contractible Reeb orbits for (M, ξ) , it is possible to eliminate all its elliptic orbits up to a given action $L > 0$ by taking a small perturbation $\varphi\alpha$ of α . Such a perturbed contact form (with some extra normalizations near the hyperbolic orbits) will be called *L-supersimple*. The elimination of elliptic orbits will be reviewed in Section 2.

The advantage of using an *L-supersimple* contact form is that, if v is an m -fold branched cover of a finite energy holomorphic map u with b simple branch points, then the Fredholm index ind , given by Equation (3.3.1), satisfies

$$\text{ind}(v) = m \text{ind}(u) + b;$$

see Lemma 3.3.2. In particular, if $\text{ind}(u) \geq 0$, then $\text{ind}(v) \geq 0$.

Remark 1.0.3. Using *L-supersimple* contact forms is mostly a matter of convenience, used to reduce the number of possible cases that we need to consider. It is expected (although not worked out in this paper) that elliptic orbits can be treated in a similar manner.

The chain groups $CC^L(M, \varphi\alpha, J)$ are generated by the good Reeb orbits of action $< L$ for $\varphi\alpha$ and the differential ∂ counts J -holomorphic maps of $\text{ind} = 1$ as usual. Here we require $(\varphi\alpha, J)$ to be an *L-supersimple pair*; see Section 3.2 for the definition. Its significance will be explained later in this section.

We need to verify the following:

- (i) $\partial^2 = 0$;
- (ii) an exact symplectic cobordism gives rise to a chain map; and

(iii) a homotopy of cobordisms gives rise to a chain homotopy.

For supersimple contact forms, (i) and (ii) are not difficult with the aid of automatic transversality techniques of Wendl [We]. The contact homology group $HC(\mathcal{D})$ is then defined as the direct limit of $HC^{L_i}(\varphi_i\alpha, J_i)$ as $L_i \rightarrow \infty$. Automatic transversality will be reviewed in Section 4, and (i) and (ii) will be proven in Section 5.

In order to prove (iii), we need to make one type of obstruction bundle calculation using the setup of Hutchings-Taubes [HT2], which takes up the rest of the paper. The prototypical gluing problem is the following (there are a few variations, but all of them can be understood in the same way, as explained in Sections 7 and 8):

Prototypical gluing problem. Let \overline{J}^τ , $\tau \in [0, 1]$, be a 1-parameter family of almost complex structures adapted to a 1-parameter family of completed exact symplectic cobordisms $(\widehat{X}^\tau, \widehat{\alpha}^\tau)$, $\tau \in [0, 1]$, and let J_\pm be the adapted¹ almost complex structures which agree with \overline{J}^τ at the positive/negative symplectization ends. Let $v_0 \cup v_1$ be a two-level SFT building, arranged from bottom to top as we go from left to right,² where:

- (C1) v_1 is a holomorphic cylinder from γ to γ'' and v_0 is a holomorphic cylinder from γ'' to γ' ;³ we assume that γ'' is negative hyperbolic and γ' is positive hyperbolic;
- (C2) v_0 maps to a cobordism $(X^{\tau_0}, \alpha^{\tau_0}, \overline{J}^{\tau_0})$ for some $\tau_0 \in (0, 1)$ and v_1 maps to a symplectization;
- (C3) $\text{ind}(v_0) = -k$, $\text{ind}(v_1) = k$, $k > 1$;
- (C4) v_0 is a k -fold unbranched cover of a transversely cut out (in a 1-parameter family) cylinder u_0 with $\text{ind}(u_0) = -1$ and v_1 is regular; and we write $v_0 = u_0 \circ \pi$, where π is the covering map.

The curves v_0 and v_1 are also equipped with asymptotic markers at the positive and negative ends; this will be described more precisely later. Let $\mathcal{M} = \mathcal{M}_{J_+}$ be the moduli space of v_1 's from γ to γ'' satisfying the above. We want to glue v_0 to \mathcal{M}/\mathbb{R} (or its compactification $\overline{\mathcal{M}}/\mathbb{R}$). See Figure 1.

By a slight modification of [HT2], there is an obstruction bundle

$$\mathcal{O} \rightarrow [R, \infty) \times \overline{\mathcal{M}/\mathbb{R}}, \quad R \gg 0,$$

whose fiber over (T, v_1) is

$$\mathcal{O}(T, v_1) = \text{Hom}(\text{Ker}((D_{v_0}^N)^*)/\mathbb{R}\langle Y \rangle, \mathbb{R}),$$

and a section \mathfrak{s} of the bundle whose zeros we are trying to count. Here we assume that v_0 is immersed for notational convenience, $D_{v_0}^N$ is the linearized normal $\overline{\partial}$ -operator of v_0 (i.e., the linearized $\overline{\partial}$ -operator projected to the normal direction), $(D_{v_0}^N)^*$ is its L^2 -adjoint, and Y is a nonzero element of $\text{coker}(D_{v_0}^N)$ which satisfies

¹More precisely, “tame”, which is defined in Section 3.1.

²This will be our usual convention.

³By a curve from γ_+ to γ_- we mean a curve which is asymptotic to γ_+ at the positive end and to γ_- at the negative end.

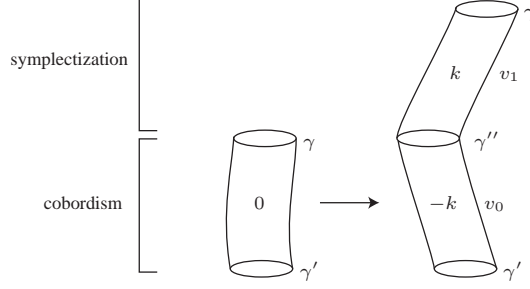


FIGURE 1. The degeneration to the limit $v_0 \cup v_1$. The numbers on the holomorphic curves represent Fredholm indices.

the following: The linearizations of the operators $\bar{\partial}_{\mathcal{T}^r}(\exp_{u_0} \xi_0)$ and $\bar{\partial}_{\mathcal{T}^r}(\exp_{v_0} \xi)$, projected to the normal direction, can be written as $D_{u_0}^N \xi_0 + (\tau - \tau_0)Y'_0$ and $D_{v_0}^N \xi + (\tau - \tau_0)Y'$, respectively, where $Y' = \pi^*Y'_0$. If Π_{u_0} and Π_{v_0} are orthogonal projections to $\ker(D_{u_0}^N)^*$ and $\ker(D_{v_0}^N)^*$, then $Y_0 = \Pi_{u_0}Y'_0$, $Y = \Pi_{v_0}Y'$, and $Y = \pi^*Y_0$.

Remark 1.0.4. In general, if v_0 is not immersed, we can replace $D_{v_0}^N$ by the full linearized $\bar{\partial}$ -operator D_{v_0} , which has isomorphic kernel and cokernel as $D_{v_0}^N$ and has Fredholm index equal to $\text{ind}(v_0)$ as given by Equation (3.3.1). For this reason we sometimes omit the superscript N from the notation.

Suppose γ'' is an $m(\gamma'')$ -fold cover of a simple orbit γ''_0 . Let $\gamma''_0 \times D^2$ be a neighborhood of γ''_0 and let $(\mathbb{R}/\mathbb{Z}) \times D^2 \rightarrow \gamma''_0 \times D^2$ be its $m(\gamma'')$ -fold cover with coordinates $(t, z = x + iy)$ such that $\{z = 0\}$ corresponds to γ'' . Also let $\mathbb{R} \times (\mathbb{R}/\mathbb{Z}) \times D^2$ be the cylinder over $(\mathbb{R}/\mathbb{Z}) \times D^2$ with coordinates $(s, t, z = x + iy)$. We parametrize \mathbb{R}/\mathbb{Z} such that the asymptotic marker of v_1 at the negative end corresponds to $t = 0$.

We consider the negative end of $v_1 \in \mathcal{M}$ and write it as a graph $\eta_1(s, t)$ over a subset of $\mathbb{R} \times (\mathbb{R}/\mathbb{Z})$. Let

$$A = -j_0 \frac{\partial}{\partial t} - S(t) : W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2) \rightarrow L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2)$$

be the asymptotic operator of γ'' , where $j_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $S(t)$ is a family of symmetric matrices. The sign convention for A is consistent with [HWZ1, HWZ2] but opposite to that of [HT1, HT2]. Asymptotic eigenfunctions will be discussed in more detail in Sections 4 and 6.

In an idealized situation, $\eta_1 \in \mathcal{M}$ admits a “Fourier expansion”

$$\eta_1(s, t) = \sum_{i \in \mathbb{Z} - \{0\}} c_i(s) e^{\lambda_i s} f_i(t),$$

where $f_i(t)$ is an eigenfunction of the asymptotic operator A corresponding to γ'' with unit L^2 -norm, λ_i is the corresponding eigenvalue,

$$(1.0.1) \quad \cdots \leq \lambda_{-2} \leq \lambda_{-1} < 0 < \lambda_1 \leq \lambda_2 \leq \cdots,$$

$\{f_i(t)\}_{i \in \mathbb{Z} - \{0\}}$ forms an orthonormal basis of $L^2(\mathbb{R}/\mathbb{Z}; \mathbb{R}^2)$, and $c_i(s)$ limits to a constant c_i as $s \rightarrow -\infty$ with $c_i = 0$ for $i < 0$. (If there are multiple eigenvalues, then we will make specific, convenient choices for $f_i(t)$.) Hence there would be an evaluation map

$$\begin{aligned} ev^k : \mathcal{M} &\rightarrow \mathbb{R}^k, \\ v_1 &\mapsto (c_1, \dots, c_k), \end{aligned}$$

and a corresponding quotient

$$\tilde{ev}^k : \mathcal{M}/\mathbb{R} \rightarrow \mathbb{R}^k/\mathbb{R}^+ \simeq S^{k-1}$$

given by \mathbb{R} -translation.

In reality, unless the $\bar{\partial}_{J_+}$ -equation is linear, i.e., of the form

$$\frac{\partial \eta}{\partial s} + j_0 \frac{\partial \eta}{\partial t} + S(t)\eta = 0$$

for curves that are close to and graphical over $\mathbb{R} \times \gamma''$, the nonlinear terms seem to interfere with the definition of the higher-order terms in the evaluation map and perhaps the best one can do is Siefring's asymptotic analysis [Si]. This now brings us to the *key reason* for using L -supersimple pairs $(\varphi\alpha, J)$: For $(\varphi\alpha, J)$ supersimple, $\bar{\partial}_J$ is linear for curves that are close to and graphical over $\mathbb{R} \times \gamma''$ and we are able to define the evaluation map; in fact $c_i(s) = c_i$ for all $s \ll 0$.

Lemma 1.0.5. *Suppose (C1)–(C4) hold. Then there exists a basis $\{\sigma_1, \dots, \sigma_k\}$ for $\ker(D_{v_0}^N)^*$, such that the positive ends of σ_i , $i = 1, \dots, k$, are of the form*

$$(1.0.2) \quad \sigma_i(s, t) = e^{-\lambda_i s} f_i(t) \quad \text{modulo } f_{k+1}, f_{k+2}, \dots$$

and $\sigma_k = Y$ modulo f_{k+1}, f_{k+2}, \dots (up to a nonzero constant multiple).

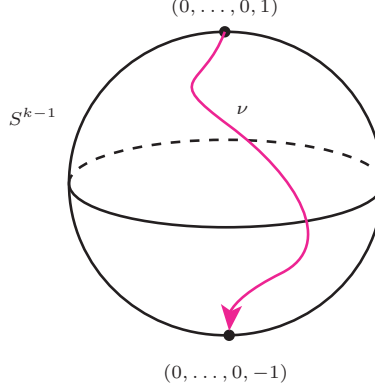
Note that $D_{v_0}^N$ is given locally by $\frac{\partial}{\partial s} - A$ and $(D_{v_0}^N)^*$ by $\frac{\partial}{\partial s} + A$, when v_0 is close to and graphical over $\mathbb{R} \times \gamma''$.

The section \mathfrak{s} of the obstruction bundle \mathcal{O} is homotopic to a section \mathfrak{s}_0 which is given by:

$$(1.0.3) \quad \mathfrak{s}_0(T, v_1)(\sigma_i) = e^{-2\lambda_i T} c_i,$$

where $ev^k(v_1) = (c_1, \dots, c_k)$, $i = 1, \dots, k-1$. In other words, on each slice $\{T\} \times \mathcal{M}/\mathbb{R}$, \mathfrak{s}_0 is more or less an evaluation map. It is important that the zeros of the homotopy \mathfrak{s}_ζ , $\zeta \in [0, 1]$, stay away from the boundary $[R, \infty) \times \partial(\mathcal{M}/\mathbb{R})$ as we go from $\mathfrak{s} = \mathfrak{s}_1$ to \mathfrak{s}_0 . This is proved in Section 8.7.

In Section 6 we prove the transversality of \tilde{ev}^k for generic J (Theorem 6.0.4). By the transversality of \tilde{ev}^k and Equation (1.0.3), the zero set $(\mathfrak{s}_0)^{-1}(0)$ is given by $[R, \infty) \times (\tilde{ev}^k)^{-1}(\{(0, \dots, 0, \pm 1)\})$. Now, let ν be a generic embedded path in S^{k-1} from $(0, \dots, 0, 1)$ to $(0, \dots, 0, -1)$. An analysis of $(\tilde{ev}^k)^{-1}(\nu)$ yields a chain homotopy term $K' \circ \partial$, where K' is part of the full chain homotopy K . This will be carried out in Section 7. \square

FIGURE 2. The path $\nu \subset S^{k-1}$.

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2. ELIMINATION OF ELLIPTIC ORBITS

Convention 2.0.1. In this paper an “orbit” or “Reeb orbit” is a closed Reeb orbit, unless stated otherwise. Also, a Reeb orbit may be a multiple cover of a simple orbit.

In what follows, M is a closed oriented 3-manifold. Given a contact form α on M , we denote its Reeb vector field by R_α . The α -action of an orbit γ is given by $\mathcal{A}_\alpha(\gamma) = \int_\gamma \alpha$. The starting point of the work is the following theorem:

Theorem 2.0.2 (Elimination of elliptic orbits). *Let α be a nondegenerate contact form for (M, ξ) . Then for any $L > 0$ and $\varepsilon > 0$ there exists a smooth function $\varphi : M \rightarrow \mathbb{R}^+$ such that:*

- (1) φ is ε -close to 1 with respect to a fixed C^1 -norm; and
- (2) all the orbits of $R_{\varphi\alpha}$ of $\varphi\alpha$ -action less than L are hyperbolic.

Moreover, we may assume that:

- (3) each positive hyperbolic orbit γ has a neighborhood $(\mathbb{R}/\mathbb{Z}) \times D_{\delta_0}^2$ with coordinates t, x, y such that
 - (a) $D_{\delta_0}^2 = \{x^2 + y^2 \leq \delta_0\}$, $\delta_0 > 0$ small;
 - (b) $\varphi\alpha = Hdt + \beta$;
 - (c) $H = c(\gamma) - \varepsilon xy$ with $c(\gamma), \varepsilon > 0$ and $c(\gamma) \gg \varepsilon$;
 - (d) $\beta = 2xdy + ydx$; and
 - (e) $\gamma = \{x = y = 0\}$.

- (4) *each negative hyperbolic orbit γ has a neighborhood $([0, 1] \times D_{\delta_0}^2)/\sim$ with coordinates t, x, y and identification $(1, x, y) \sim (0, -x, -y)$ satisfying (a)–(e).*

Proof. Essentially given in the proof of [CGH1, Theorem 2.5.2]. \square

Definition 2.0.3. Let $L > 0$. Then a contact form α is:

- (1) *L -nondegenerate* if all the Reeb orbits of action $< L$ are nondegenerate;
- (2) *L -supersimple* if α is L -nondegenerate and all the Reeb orbits of action $< L$ are hyperbolic with neighborhoods that are given in Theorem 2.0.2(3) and (4);
- (3) *L -noncontractible* if α is L -nondegenerate and has no contractible Reeb orbits of action $< L$; and
- (4) *L -monotone* if α is L -nondegenerate and has no contractible Reeb orbits of action $< L$ and Conley-Zehnder index ≤ 2 with respect to any bounding disk.

Corollary 2.0.4. *Let α be a nondegenerate contact form for (M, ξ) and let L_1, L_2, \dots and $\varepsilon_1, \varepsilon_2, \dots$ be positive increasing sequences such that $\lim_i L_i = \infty$ and $\lim_i \varepsilon_i = \varepsilon$ for some $\varepsilon > 0$. Then there exists a sequence $\varphi_1, \varphi_2, \dots$ of functions $\varphi_i : M \rightarrow \mathbb{R}^+$ where φ_i is ε_i -close to 1 and $\varphi_i \alpha$ is L_i -supersimple.⁴*

3. ALMOST COMPLEX STRUCTURES AND MODULI SPACES

Let α be a contact form and R_α be its Reeb vector field. We write $\gamma = (\gamma_1, \dots, \gamma_l)$ for some ordered tuple of (not necessarily simple) Reeb orbits of R_α .

Warning 3.0.1. Unlike the case of embedded contact homology [Hu], γ_i may be multiply-covered.

3.1. α -tame almost complex structures. The almost complex structures on $\mathbb{R} \times M$ that we use are slight variants of almost complex structures which are commonly known as “adapted to” or “compatible with” a contact form α for the contact manifold (M, ξ) .

Definition 3.1.1. An almost complex structure J on $\mathbb{R} \times M$ with coordinates (s, x) is α -tame if

- J is \mathbb{R} -invariant,
- $J(\partial_s) = g(x)R_\alpha$, where g is a positive function on M , and
- $J(\xi') = \xi'$ for some oriented 2-plane field ξ' of M on which $d\alpha$ is symplectic and $d\alpha(v, Jv) > 0$ for nonzero $v \in \xi'$.⁵

Note that $d\alpha|_{\xi'}$ being symplectic is equivalent to R_α being positively transverse to ξ' .

The α -tameness of J is sufficient to guarantee that the moduli space of “finite energy” J -holomorphic curves satisfies SFT compactness; see Section 3.4.

⁴Unfortunately, in general the limit $\lim_i \varphi_i$ is not a smooth function.

⁵Note that we are not requiring $\xi' = \xi$.

The space of all α -tame J in the class C^∞ will be denoted by $\mathcal{J}(\alpha)$ or by \mathcal{J} if α is understood.

3.2. L -supersimple pairs. When $\varphi\alpha$ is L -supersimple, we choose a $\varphi\alpha$ -tame J which satisfies the following on each neighborhood $(\mathbb{R}/\mathbb{Z}) \times D_{\delta_0}^2$ of a positive hyperbolic orbit of action $< L$ given in Theorem 2.0.2 (3) (and an analogous condition for each neighborhood of a negative hyperbolic orbit):

- (J1) $\xi' = \xi$ on $(\mathbb{R}/\mathbb{Z}) \times (D_{\delta_0}^2 - D_{2\delta_0/3}^2)$, $\xi' = TD_{\delta_0}^2$ on $(\mathbb{R}/\mathbb{Z}) \times D_{\delta_0/3}^2$, and ξ' is close to ξ in between;
- (J2) $J : \partial_s \mapsto \partial_t + X_H = \partial_t - \varepsilon(x\partial_x - y\partial_y) = HR_{\varphi\alpha}$ and $\partial_x \mapsto \partial_y$ on $(\mathbb{R}/\mathbb{Z}) \times D_{\delta_0/3}^2$.

The subset of such J will be denoted by $\mathcal{J}_{\star L}(\varphi\alpha)$ or $\mathcal{J}_\star(\varphi\alpha)$ when L is understood. Finally, a pair $(\varphi\alpha, J)$ consisting of an L -supersimple $\varphi\alpha$ and a tame J satisfying the above will be called an *L -supersimple pair*.

3.3. Moduli spaces. Let α be a nondegenerate contact form on M and let J be an α -tame almost complex structure.

Definition 3.3.1. Let $\gamma_+ = (\gamma_{+,1}, \dots, \gamma_{+,l_+})$ and $\gamma_- = (\gamma_{-,1}, \dots, \gamma_{-,l_-})$ be ordered tuples for α and let J be α -tame. We say a J -holomorphic map $u : \dot{F} \rightarrow \mathbb{R} \times M$ is *from γ_+ to γ_-* if it is asymptotic to γ_+ at the positive end and to γ_- at the negative end. Here F is a closed Riemann surface, $\mathbf{p} = \mathbf{p}_+ \sqcup \mathbf{p}_-$ is a finite ordered subset of F (whose points are called *punctures*), $\dot{F} = F - \mathbf{p}$, \mathbf{p}_+ corresponds to γ_+ in order, and \mathbf{p}_- corresponds to γ_- in order.

Fredholm index. The Fredholm index of u is given by:

$$(3.3.1) \quad \text{ind}(u) = -\chi(\dot{F}) + \mu_\tau(\gamma_+) - \mu_\tau(\gamma_-) + 2c_1(u^*\xi, \tau),$$

where τ is a framing for ξ defined along γ_+ and γ_- , μ_τ is the Conley-Zehnder index with respect to τ , $\mu_\tau(\gamma_\pm) := \sum_i \mu_\tau(\gamma_{\pm,i})$, and $c_1(u^*\xi, \tau)$ is the first Chern class of $u^*\xi$ with respect to τ . (Here we are viewing ξ as a 2-plane field of M and also $\mathbb{R} \times M$.) Here ind keeps track of both the variations of complex structures of \dot{F} and the infinitesimal automorphisms of \dot{F} .

The most important aspect of working with L -supersimple α is the following:

Lemma 3.3.2. *Let u be a J -holomorphic map from γ_+ to γ_- , where all the orbits of γ_+ and γ_- are hyperbolic. If v is a k -fold branched cover of u with total branching multiplicity b , then $\text{ind}(v) = k \text{ind}(u) + b$.*

Here b is the sum over all the branch points of the order of multiplicity minus one; in particular, if all the branch points are double points, then b is the number of branch points.

Proof. Follows immediately from observing that the Conley-Zehnder indices of hyperbolic orbits behave multiplicatively when we take multiple covers of Reeb orbits. \square

Definition of $\mathcal{M}_J(\gamma_+, \gamma_-)$. Pick a point x_γ on each simple Reeb orbit γ of R_α . Let (u, \mathbf{r}) be a pair consisting of a J -holomorphic map $u : \dot{F} \rightarrow \mathbb{R} \times M$ from γ_+ to γ_- and an ordered set $\mathbf{r} = \mathbf{r}_+ \sqcup \mathbf{r}_-$ of asymptotic markers, where

$$\mathbf{r}_+ = (r_{+,1}, \dots, r_{+,l_+}) \quad \text{and} \quad \mathbf{r}_- = (r_{-,1}, \dots, r_{-,l_-})$$

correspond to punctures \mathbf{p}_+ and \mathbf{p}_- , the marker $r_{\pm,i}$ is “mapped to” $x_{\gamma_{\pm,i}^s}$, and $\gamma_{\pm,i}^s$ is the simple orbit corresponding to $\gamma_{\pm,i}$. Here an *asymptotic marker* at a puncture z of F is an element of $(T_z F - \{0\})/\mathbb{R}^+$. The moduli space $\mathcal{M}_J(\gamma_+, \gamma_-)$ is the space of (u, \mathbf{r}) , modulo biholomorphisms of the domain that take positive markers to positive markers and negative markers to negative markers.

For convenience we will suppress \mathbf{r} from (u, \mathbf{r}) when there is no confusion.

If $\gamma_\pm = (\gamma_{\pm,1})$, then we also write $\gamma_\pm = \gamma_\pm$ and $\mathcal{M}_J(\gamma_+, \gamma_-) = \mathcal{M}_J(\gamma_+, \gamma_-)$; similarly, if $\mathbf{r}_\pm = (r_{\pm,1})$, then we also write $\mathbf{r}_\pm = r_\pm$. We write $\mathcal{M}_J^*(\gamma_+, \gamma_-)$ to denote the subset of $\mathcal{M}_J(\gamma_+, \gamma_-)$ satisfying $*$. In particular, $\text{ind} = k$ means “Fredholm index k ”, s means “simple (= non-multiply-covered)”, A means “in the homology class $A \in H_2(M; \mathbb{Z})$ ”, sing means “singular”, i.e., non-immersed, cyl means we only count cylinders, and δ_0 indicates the radius of D^2 in (J1) and (J2).

A generic $J \in \mathcal{J}$ is *regular*, i.e., the moduli spaces $\mathcal{M}_J^s(\gamma_+, \gamma_-)$ are transversely cut out for all γ_+, γ_- . Let $\mathcal{J}^{\text{reg}} \subset \mathcal{J}$ be the subset of regular J and let $\mathcal{J}^{<L, \text{reg}} \subset \mathcal{J}$ be the subset of J for which $\mathcal{M}_J^s(\gamma_+, \gamma_-)$ is transversely cut out for all γ_+, γ_- with action $< L$. The space $\mathcal{J}_*^{<L, \text{reg}} = \mathcal{J}_{*L}^{<L, \text{reg}}$ of regular L -supersimple almost complex structures (with respect to a fixed α) is defined similarly.

Remark 3.3.3. We sometimes say that a curve is *regular* if it is transversely cut out. This should not be confused with a curve being singular, which means the curve is not an immersion.

Finally, we use the coherent orientation system for all $\mathcal{M}_J(\gamma_+, \gamma_-)$ as described in Bourgeois-Mohnke [BM].

3.4. Compactness. In this subsection we show that the moduli space $\mathcal{M}_J(\gamma_+, \gamma_-)$ can be compactified by adding holomorphic buildings in the SFT sense; see Section 8 of [BEHWZ].

Theorem 3.4.1. *An α -tame almost complex structure J satisfies the usual SFT compactness properties as described in [BEHWZ].*

Proof. The proof follows the line given by [Ho, HWZ1, BEHWZ] with the following modifications.

Let $u : \dot{F} \rightarrow \mathbb{R} \times M$ be a J -holomorphic map. The $d\alpha$ -energy of u is defined as usual by

$$E_{d\alpha}(u) = \int_{\dot{F}} u^* d\alpha$$

and the α -energy of u is defined slightly differently by

$$E_\alpha(u) = \sup_{\phi \in \mathcal{C}_K} \int_{\dot{F}} u^*(\phi'(s) ds \wedge \alpha),$$

where \mathcal{C}_K is the set of smooth functions $\phi : \mathbb{R} \rightarrow [1, 2]$ such that $\phi(s) = 1$ for $s \ll 0$, $\phi(s) = 2$ for $s \gg 0$, and $0 \leq \phi'(s) \leq K$ for all s .

Remark 3.4.2 (Energy bounds). Let u be a J -holomorphic map from γ_+ to γ_- . While it is possible for $u^*(\phi'(s)ds \wedge \alpha)$ to be negative at certain points due to the terms $a\alpha(JX)$ and $b\alpha(X)$ that appear in Equation (3.4.2) below, the α -energy $E_\alpha(u)$ is bounded above and below: Write

$$\int_u \phi'(s)ds \wedge \alpha = \int_u d(\phi\alpha) - \int_u \phi d\alpha.$$

The right-hand side can be bounded since $\int_u d(\phi\alpha) = 2\mathcal{A}_\alpha(\gamma_+) - \mathcal{A}_\alpha(\gamma_-)$, $\int_u \phi d\alpha$ is positive and satisfies $\int_u d\alpha \leq \int_u \phi d\alpha \leq \int_u 2d\alpha$, and $\int_u d\alpha = \mathcal{A}_\alpha(\gamma_+) - \mathcal{A}_\alpha(\gamma_-)$. Hence

$$(3.4.1) \quad \mathcal{A}_\alpha(\gamma_-) \leq E_\alpha(u) \leq \mathcal{A}_\alpha(\gamma_+).$$

Lemma 3.4.3. *There exists $K > 0$ small such that, all $\phi \in \mathcal{C}_K$, $d(\phi\alpha)(v, Jv) \geq 0$ for all v and $d(\phi\alpha)(v, Jv) > 0$ for all nonzero v on the region $\phi'(s) > 0$.*

Proof of Lemma 3.4.3. Write $v = X + a\partial_s + bJ\partial_s = X + a\partial_s + bgR_\alpha$, where $X \in \xi'$. Then we have $Jv = JX - b\partial_s + agR_\alpha$ and

$$(3.4.2) \quad \begin{aligned} d(\phi\alpha)(v, Jv) &= \phi'(s)(ds \wedge \alpha)(v, Jv) + \phi d\alpha(v, Jv) \\ &= \phi'(s)(a^2g + a\alpha(JX) + b^2g + b\alpha(X)) + \phi d\alpha(X, JX). \end{aligned}$$

Here $d\alpha(X, JX) > 0$ if $X \neq 0$. Also note that $d\alpha(X, JX)$ is bounded below by a positive constant times $|X|^2$; and $|\alpha(X)|$ and $|\alpha(JX)|$ are bounded above by a positive constant times $|X|$, where the norm $|\cdot|$ is measured with respect to some fixed Riemannian metric on M . Since $\phi \geq 1$, by taking $0 \leq \phi'(s) < K$ for K small, we obtain $d(\phi\alpha)(v, Jv) \geq 0$ for all v . Similarly, $d(\phi\alpha)(v, Jv) > 0$ for all nonzero v on the region $\phi'(s) > 0$. [The point is that, if we fix a constant k , then $\varepsilon a^2 \pm \varepsilon kac + c^2 > 0$ for all $(a, c) \neq (0, 0)$ and sufficiently small $\varepsilon > 0$: First observe that

$$(\sqrt{\varepsilon}a \pm c)^2 = \varepsilon a^2 \pm 2\sqrt{\varepsilon}ac + c^2 \geq 0.$$

Then $\varepsilon a^2 + c^2 \geq |2\sqrt{\varepsilon}ac| > |\varepsilon kac|$ for sufficiently small $\varepsilon > 0$, provided $a \neq 0$ and $c \neq 0$. On the other hand, if $a = 0$ and $c \neq 0$ or $a \neq 0$ and $c = 0$, then $\varepsilon a^2 \pm \varepsilon kac + c^2 > 0$ is immediate, provided $\varepsilon > 0$.] \square

Returning to the proof of Theorem 3.4.1, let $u : \dot{F} \rightarrow \mathbb{R} \times M$ be a J -holomorphic map with *finite energy*, i.e., $E_{d\alpha}(u)$ and $E_\alpha(u)$ are finite.

We will retrace some key steps of Hofer [Ho]. If $E_{d\alpha}(u) = 0$, then u maps to a cylinder over a (not necessarily closed) Reeb trajectory γ . If $\dot{F} = \mathbb{C}$ in addition, then the little Picard theorem applied to $\tilde{u} : \mathbb{C} \rightarrow \mathbb{R} \times \tilde{\gamma} \subset \mathbb{R} \times M$ implies that u is a constant map. Here $\tilde{\gamma}$ is the universal cover of γ and \tilde{u} is the lift of $u : \mathbb{C} \rightarrow \mathbb{R} \times \gamma$. Note that using a smaller class \mathcal{C}_K of “slow growth” test functions does not alter this argument or any other argument where E_α is used.

Next, a finite energy u does not have gradient blow-up at the ends of \dot{F} , since otherwise we can construct a nonconstant finite energy plane with $E_{d\alpha} = 0$ using the “Hofer lemma” and the taming of J by $d(\phi\alpha)$, $\phi \in \mathcal{C}_K$, on compact regions. (Here the gradient is measured with respect to some \mathbb{R} -invariant Riemannian metric on $\mathbb{R} \times M$ and a cylindrical metric on the ends of u .) The gradient bound implies that if $E_{d\alpha}(u) = 0$, $\dot{F} = \mathbb{R} \times S^1$, and u is nonconstant, then u maps to a cylinder over a closed orbit γ and

$$\begin{aligned} u : \dot{F} &\rightarrow \mathbb{R} \times \gamma = \mathbb{R} \times (\mathbb{R}/\mathbb{Z}), \\ (s', t') &\mapsto (s' + c, mt' + d), \end{aligned}$$

where $c, d \in \mathbb{R}$ and $m \in \mathbb{Z}^+$ (this follows from the fact that an entire holomorphic function with bounded derivative is a linear map). Finally, the gradient bound, the E_α -bound, and the taming of J together imply that $u|_{\mathcal{E}}$ (here \mathcal{E} is an end) converges either to a cylinder over a Reeb orbit or to a removable singularity.

Having controlled the behavior of holomorphic curves at the ends (and analogously on the necks), the rest of the SFT compactness argument then carries over. \square

4. AUTOMATIC TRANSVERSALITY AND APPLICATIONS

4.1. Calculation of asymptotic eigenvalues and eigenfunctions. For the simplicity of notations, in this section we only look at the orbits of action 1. Let $S^1 = \mathbb{R}/\mathbb{Z}$ with coordinate t . In this subsection we explicitly calculate the eigenvalues and eigenfunctions for certain asymptotic operators

$$\begin{aligned} A : W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2) &\rightarrow L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2), \\ A &= -j_0 \frac{\partial}{\partial t} - S(t), \end{aligned}$$

where $j_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $S(t)$ is a family of symmetric matrices. This is mostly for reference (especially the elliptic case, which is never used) and is completely standard.

4.1.1. Elliptic case. Let $S(t) = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}$, where $0 < \varepsilon < 2\pi$. If we view $\mathbb{R}^2 = \mathbb{C}$, then J is multiplication by i and $S(t)$ is multiplication by ε . If f is an eigenfunction of A with eigenvalue λ , then

$$Af = -i \frac{\partial f}{\partial t} - \varepsilon f = \lambda f$$

and $f'(t) = i(\lambda + \varepsilon)f$. Hence $\lambda + \varepsilon = 2\pi n$ for $n \in \mathbb{Z}$ and $\lambda = 2\pi n - \varepsilon$; the corresponding asymptotic eigenfunction is $f(t) = ce^{2\pi int}$, $c \in \mathbb{C}$. We write

$$\begin{aligned} f_{2n-1}(t) &= e^{2\pi int}, f_{2n} = ie^{2\pi int}, \\ \lambda_{2n-1} &= \lambda_{2n} = 2\pi n - \varepsilon, \end{aligned}$$

for $n \geq 1$ and

$$f_{2n-2}(t) = e^{2\pi int}, f_{2n-1} = ie^{2\pi int},$$

$$\lambda_{2n-2} = \lambda_{2n-1} = 2\pi n - \varepsilon,$$

for $n \leq 0$.⁶ Note that $\lambda_{2n-1} = \lambda_{2n}$ is a multiple eigenvalue and the corresponding eigenspace is a complex vector space $\mathbb{C}\langle f_{2n-1}(t) \rangle$.

4.1.2. *Positive hyperbolic case.* Let $S(t) = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}$, where ε is a small positive number. Then $Af = \lambda f$ can be written as:

$$f'(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda & \varepsilon \\ \varepsilon & \lambda \end{pmatrix} f = \begin{pmatrix} -\varepsilon & -\lambda \\ \lambda & \varepsilon \end{pmatrix} f.$$

We diagonalize the matrix $P = \begin{pmatrix} -\varepsilon & -\lambda \\ \lambda & \varepsilon \end{pmatrix}$: We solve for μ in

$$\det \begin{pmatrix} -\varepsilon - \mu & -\lambda \\ \lambda & \varepsilon - \mu \end{pmatrix} = \mu^2 + (\lambda^2 - \varepsilon^2) = 0.$$

Case $|\lambda| > \varepsilon$. The eigenvalues of P are $\mu = \pm i\sqrt{\lambda^2 - \varepsilon^2}$ and the eigenvectors are

$$v = \begin{pmatrix} \varepsilon - i\sqrt{\lambda^2 - \varepsilon^2} \\ -\lambda \end{pmatrix}, \quad \bar{v} = \begin{pmatrix} \varepsilon + i\sqrt{\lambda^2 - \varepsilon^2} \\ -\lambda \end{pmatrix}.$$

In order for $f(0) = f(1)$, we require $\sqrt{\lambda^2 - \varepsilon^2} = 2\pi n$. Here $n \neq 0$ since $|\lambda| > \varepsilon$. Hence $\lambda = \pm\sqrt{(2\pi n)^2 + \varepsilon^2}$. Then f is the real or imaginary part of $e^{i\sqrt{\lambda^2 - \varepsilon^2}t}v = e^{i2\pi nt}v$. Then for $n > 0$ we obtain

$$(4.1.1) \quad f_{\pm 2n}(t) = \begin{pmatrix} \varepsilon \cos(2\pi nt) + 2\pi n \sin(2\pi nt) \\ \mp \sqrt{(2\pi n)^2 + \varepsilon^2} \cos(2\pi nt) \end{pmatrix},$$

$$(4.1.2) \quad f_{\pm(2n+1)}(t) = \begin{pmatrix} -2\pi n \cos(2\pi nt) + \varepsilon \sin(2\pi nt) \\ \mp \sqrt{(2\pi n)^2 + \varepsilon^2} \sin(2\pi nt) \end{pmatrix},$$

$$\lambda_{\pm 2n} = \lambda_{\pm(2n+1)} = \mp \sqrt{(2\pi n)^2 + \varepsilon^2}.$$

In particular, we have multiple eigenvalues when $\lambda \neq \pm\varepsilon$.

Case $|\lambda| \leq \varepsilon$. The eigenvalues are real, and in order for $f(0) = f(1)$ we require $\lambda = \pm\varepsilon$. We obtain

$$f_{-1}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad f_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$\lambda_{-1} = -\varepsilon, \quad \lambda_1 = \varepsilon.$$

⁶We choose this slightly strange numbering so it is consistent with Equation (1.0.1).

4.1.3. Negative hyperbolic case. We identify a neighborhood of a negative hyperbolic orbit with $((\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2)/\sim$, where (t, x) are coordinates on $[0, 1] \times \mathbb{R}^2$ and $(1, x) \sim (0, -x)$. With respect to these coordinates we write $A = -j_0 \frac{\partial}{\partial t} - S(t)$ with $S(t) = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}$. The calculations are similar to the positive hyperbolic case and for $n > 1$ we obtain

$$(4.1.3) \quad f_{\pm(2n-1)}(t) = \begin{pmatrix} \varepsilon \cos((2n-1)\pi t) + (2n-1)\pi \sin((2n-1)\pi t) \\ \mp \sqrt{((2n-1)\pi)^2 + \varepsilon^2} \cos((2n-1)\pi t) \end{pmatrix},$$

$$(4.1.4) \quad f_{\pm 2n}(t) = \begin{pmatrix} -(2n-1)\pi \cos((2n-1)\pi t) + \varepsilon \sin((2n-1)\pi t) \\ \mp \sqrt{((2n-1)\pi)^2 + \varepsilon^2} \sin((2n-1)\pi t) \end{pmatrix},$$

$$\lambda_{\pm(2n-1)} = \lambda_{\pm 2n} = \mp \sqrt{((2n-1)\pi)^2 + \varepsilon^2}.$$

4.2. Automatic transversality. In this subsection we summarize the parts of the proof of the automatic transversality theorem of Wendl [We] which are used later. Automatic transversality was originally due to Gromov [Gr] and worked out carefully by Hofer-Lizan-Sikorav [HLS] for closed J -holomorphic curves. In this section $J \in \mathcal{J}(\alpha)$.

A Reeb orbit is *even* (resp. *odd*) if its Conley-Zehnder index is even (resp. odd) (with respect to any trivialization τ). Note that a Reeb orbit is even if and only if it is positive hyperbolic. Given a J -holomorphic curve u , let $\#\Gamma_0(u)$ be the number of ends that limit to an even orbit and $\#\Gamma_1(u)$ be the number of ends that limit to an odd orbit.

Theorem 4.2.1 (Automatic transversality). *Let $u : \dot{F} \rightarrow \mathbb{R} \times M$ be an element of $\mathcal{M}_J(\gamma, \gamma')$. If u is an immersion, then u is regular if*

$$(4.2.1) \quad \text{ind}(u) > 2g(F) - 2 + \#\Gamma_0(u).$$

Proof. Let $u \in \mathcal{M}_J(\gamma, \gamma')$ be an immersion and let $N \rightarrow \dot{F}$ be a normal bundle to $u(\dot{F}) \subset \mathbb{R} \times M$ such that $N = TD_{\delta'}^2$ on the ends of u for $\delta' > 0$ small. Here each Reeb orbit has a neighborhood of the form $(\mathbb{R}/\mathbb{Z}) \times D_{\delta_0}^2$ as described in Theorem 2.0.2. Let

$$D_u^N : W^{k+1,p}(\dot{F}, N) \rightarrow W^{k,p}(\dot{F}, \Lambda^{0,1} T^* \dot{F} \otimes_J N)$$

be the linearized normal $\bar{\partial}$ -operator for u and $(D_u^N)^*$ be its adjoint. The Fredholm index of D_u^N is equal to the Fredholm index $\text{ind}(u)$ of the full linearized $\bar{\partial}$ -operator. We may suppress the upper index “ N ” from D_u^N in latter part of this paper. Since $\text{coker } D_u^N \simeq \ker(D_u^N)^*$, it suffices to show $\ker(D_u^N)^* = 0$ in order to prove Theorem 4.2.1.

Let τ be a trivialization of ξ along γ and γ' . Given a nontrivial element $\eta \in \ker D_u^N$, we define the *winding number* $\text{wind}_\tau(\eta, p)$ of η near a puncture p of F to be the winding number of the leading asymptotic eigenfunction of η (projected to $TD_{\delta'}^2$) with respect to τ , viewed as a trivialization of $TD_{\delta'}^2$. We also define the *total winding number*

$$(4.2.2) \quad \text{wind}_\tau(\eta) = \sum_{p_+ \in \mathbf{p}_+} \text{wind}_\tau(\eta, p_+) - \sum_{p_- \in \mathbf{p}_-} \text{wind}_\tau(\eta, p_-).$$

For each positive puncture p that limits to γ , $2 \operatorname{wind}_\tau(\eta, p) \leq \mu_\tau(\gamma)$ and equality is possible if and only if γ is even; similarly, for each negative puncture that limits to γ , $2 \operatorname{wind}_\tau(\eta, p) \geq \mu_\tau(\gamma)$ and equality is possible if and only if γ is even. Hence

$$(4.2.3) \quad \begin{aligned} 2 \operatorname{wind}_\tau(\eta) &\leq \mu_\tau(\gamma) - \mu_\tau(\gamma') - \#\Gamma_1(u) \\ &= \mu_\tau(\gamma) - \mu_\tau(\gamma') + \#\Gamma_0(u) - k, \end{aligned}$$

where k is the total number of punctures.

The intersection number of η with the zero section of N is given by $\operatorname{wind}_\tau(\eta) + c_1(N, \tau)$ and the positivity of intersections in dimension 4 implies that:

Claim 4.2.2. *If $\eta \in \ker D_u^N$ is nonzero, then $\operatorname{wind}_\tau(\eta) + c_1(N, \tau) \geq 0$.*

Claim 4.2.3. *If*

$$(4.2.4) \quad 2c_1(N, \tau) + \mu_\tau(\gamma) - \mu_\tau(\gamma') + \#\Gamma_0(u) - k < 0,$$

then $\ker D_u^N = 0$.

Similarly, if we replace N by $\Lambda^{0,1}T^*\dot{F} \otimes N$ and note that the leading term of $(D_u^N)^*$ is the ∂ -operator, we obtain:

Claim 4.2.4. *If $\zeta \in \ker(D_u^N)^*$ is nonzero, then*

$$(4.2.5) \quad 2 \operatorname{wind}_\tau(\zeta) \geq \mu_\tau(\gamma) - \mu_\tau(\gamma') + \#\Gamma_1(u),$$

$$(4.2.6) \quad \operatorname{wind}_\tau(\zeta) + c_1(\Lambda^{0,1}T^*\dot{F} \otimes N, (ds - idt) \otimes \tau) \leq 0,$$

where $ds - idt$ is the trivialization of $\Lambda^{0,1}T^\dot{F}$ on the ends of u induced by $ds - idt$. Here (s, t) are the first two coordinates of $\mathbb{R} \times (\mathbb{R}/\mathbb{Z}) \times D_{\delta_0}^2$.*

Claim 4.2.5. *If*

$$(4.2.7) \quad 2c_1(\Lambda^{0,1}T^*\dot{F} \otimes N, (ds - idt) \otimes \tau) + \mu_\tau(\gamma) - \mu_\tau(\gamma') + k - \#\Gamma_0(u) > 0,$$

then $\ker(D_u^N)^ = 0$.*

Since

$$\chi(\dot{F}) = 2 - 2g(\dot{F}) - k = c_1(T\dot{F}, ds - idt) = c_1(\Lambda^{0,1}T^*\dot{F}, ds - idt)$$

and

$$c_1(\Lambda^{0,1}T^*\dot{F} \otimes N, (ds - idt) \otimes \tau) = c_1(\xi, \tau),$$

Inequality (4.2.7) can be rephrased as:

$$\begin{aligned} \operatorname{ind}(u) &= -\chi(\dot{F}) + 2c_1(\xi, \tau) + \mu_\tau(\gamma) - \mu_\tau(\gamma') \\ &= -\chi(\dot{F}) + 2c_1(\Lambda^{0,1}T^*\dot{F} \otimes N, (ds - idt) \otimes \tau) + \mu_\tau(\gamma) - \mu_\tau(\gamma') \\ &> -\chi(\dot{F}) + \#\Gamma_0(u) - k = 2g(\dot{F}) - 2 + \#\Gamma_0(u), \end{aligned}$$

which implies the theorem. \square

The following theorem, due to Hutchings-Taubes [HT2, Theorem 4.1] will also be used frequently in conjunction with Theorem 4.2.1.

Theorem 4.2.6. *If J is generic, then for each γ, γ' , the set of non-immersed $u \in \mathcal{M}_J^s(\gamma, \gamma')$ has real codimension at least two.*

Remark 4.2.7. [HT2, Theorem 4.1] states a less general theorem, but its proof gives Theorem 4.2.6.

4.3. Proof of Lemma 1.0.5. As an application of the automatic transversality technique, we prove the following lemmas, which are slight strengthenings of Lemma 1.0.5 and which will be used in the proof of the chain homotopy.

We are assuming (C1)–(C4). As before, at the positive end γ'' of v_1 , let $f_i(t)$, $i \in \mathbb{Z} - \{0\}$, be the eigenfunctions of the asymptotic operator such that

(D1) the corresponding eigenvalues λ_i satisfy

$$\cdots \leq \lambda_{-2} \leq \lambda_{-1} < 0 < \lambda_1 \leq \lambda_2 \leq \cdots,$$

(D2) $\{f_i(t)\}_{i \in \mathbb{Z} - \{0\}}$ forms an orthonormal basis of $L^2(\mathbb{R}/\mathbb{Z}; \mathbb{R}^2)$.

At the negative end γ' of v_0 , let $g_i(t)$, $i \in \mathbb{Z} - \{0\}$, be the eigenfunctions of the asymptotic operator such that

(E1) the corresponding eigenvalues λ'_i satisfy

$$\cdots \leq \lambda'_{-2} \leq \lambda'_{-1} < 0 < \lambda'_1 \leq \lambda'_2 \leq \cdots,$$

(E2) $\{g_i(t)\}_{i \in \mathbb{Z} - \{0\}}$ forms an orthonormal basis of $L^2(\mathbb{R}/\mathbb{Z}; \mathbb{R}^2)$.

Lemma 4.3.1. *Suppose (C1)–(C4) hold. Then there exists a basis $\{\sigma_1, \dots, \sigma_k\}$ for $\ker(D_{v_0}^N)^*$ such that the following hold:*

(1) *The positive ends of σ_i , $i = 1, \dots, k$, are of the form*

$$\sigma_i(s, t) = e^{-\lambda_i s} f_i(t) \quad \text{modulo } f_{k+1}, f_{k+2}, \dots$$

(2) *The negative ends of σ_i , $i = 1, \dots, k$, span $\mathbb{R}\langle e^{-\lambda'_j s} g_j(t) \rangle_{j=-k}^{-1}$ modulo $g_{-k-1}, g_{-k-2}, \dots$.*

(3) *The negative end of σ_i , $i = 1, \dots, k$ projects nontrivially to*

$$\mathbb{R}\langle e^{-\lambda'_j s} g_j(t) \rangle_{j=-k+i-2}^{-1}$$

if $\lambda'_{-k+i-2} = \lambda'_{-k+i-1}$ and to

$$\mathbb{R}\langle e^{-\lambda'_j s} g_j(t) \rangle_{j=-k+i-1}^{-1}$$

if $\lambda'_{-k+i-2} < \lambda'_{-k+i-1}$.

(4) *The negative end of σ_k is of the form*

$$\sigma_k(s, t) = d_{k,-1} e^{-\lambda'_{-1} s} g_{-1}(t) \quad \text{modulo } g_{-k-1}, g_{-k-2}, \dots,$$

where $d_{k,-1}$ is nonzero.

(5) $\sigma_k = Y$, up to a nonzero constant multiple.

Proof. (1) We consider the linearized normal $\bar{\partial}$ -operators

$$D_{v_0}^{\delta} = D_{v_0}^{N, \delta} : W_{\delta}^{k+1, p}(\dot{F}, N) \rightarrow W_{\delta}^{k+1, p}(\dot{F}, \Lambda^{0,1} T^* \dot{F} \otimes N),$$

where $N \rightarrow \dot{F}$ is the normal bundle to $v_0 : \dot{F} \rightarrow \widehat{X}^{\tau_0}$ and we are using weights $\chi_\delta(s)$, i.e., $\zeta \in W_\delta^{k+1,p}$ if and only if $\chi_\delta(s)\zeta \in W^{k+1,p}$. Here $\chi_\delta : \mathbb{R} \rightarrow \mathbb{R}^{>0}$ is a smooth function such that $\chi_\delta(s) = 1$ for $s \leq 0$ and $\chi_\delta(s) = e^{-\delta s}$ for $s \gg 0$.

Suppose k is odd; the case of k even is similar (but slightly harder) and will be omitted. By the asymptotic eigenfunction calculations from Section 4.1.3,

$$\text{ind}(D_{v_0}^0) = \text{ind}(D_{v_0}^{\delta_0}) = -k, \quad \text{ind}(D_{v_0}^{\delta_1}) = -k+2, \quad \dots, \quad \text{ind}(D_{v_0}^{\delta_{\lfloor k/2 \rfloor}}) = -1,$$

where $\delta_j = 2\pi j$. By Claim 4.2.2,

$$\ker(D_{v_0}^{\delta_0}) = \dots = \ker(D_{v_0}^{\delta_{\lfloor k/2 \rfloor}}) = 0.$$

Hence we obtain

$$\ker((D_{v_0}^{\delta_0})^*) = k, \quad \ker((D_{v_0}^{\delta_1})^*) = k-2, \quad \dots, \quad \ker((D_{v_0}^{\delta_{\lfloor k/2 \rfloor}})^*) = 1.$$

Now consider $\sigma_1, \sigma_2 \in \ker((D_{v_0}^{\delta_0})^*)$, whose projections span

$$\ker((D_{v_0}^{\delta_0})^*) / \ker((D_{v_0}^{\delta_1})^*).$$

For $i = 1, 2$, let us write

$$\sigma_i = c_1(\sigma_i)e^{-\lambda_1 s}f_1(t) + c_2(\sigma_i)e^{-\lambda_2 s}f_2(t) \quad \text{modulo } f_3, f_4, \dots$$

at the positive end. Then $(c_1(\sigma_1), c_2(\sigma_1))$ and $(c_1(\sigma_2), c_2(\sigma_2))$ are linearly independent, since otherwise a nontrivial linear combination of σ_1 and σ_2 will be in $\ker((D_{v_0}^{\delta_1})^*)$ and this contradicts the fact that $\ker((D_{v_0}^{\delta_0})^*) / \ker((D_{v_0}^{\delta_1})^*)$ is 2-dimensional. By induction, row reduction, and possibly renaming the σ_i , we eventually obtain the basis $\{\sigma_1, \dots, \sigma_k\}$ which satisfies Equation (1.0.2).

(2) follows from the argument of (1).

(3) follows from Claim 4.2.4.

(4), (5) By the regularity of u_0 in a 1-parameter family $\overline{\mathcal{J}}^\tau$, $\ker((D_{u_0}^N)^*)$ is 1-dimensional and is generated by Y_0 which comes from the variation of $\overline{\mathcal{J}}^\tau$. Choose a trivialization $\tilde{\tau}$ along γ'' and γ' so that $\mu_{\tilde{\tau}}(\gamma'') = -1$ and $\mu_{\tilde{\tau}}(\gamma') = 0$. By Claim 4.2.4, Y_0 has $\text{wind}_{\tilde{\tau}} = 0$ at both ends. Hence its pullback $Y = \pi^*Y_0$ to $\ker((D_{v_0}^N)^*)$ also has $\text{wind}_{\tilde{\tau}} = 0$ at both ends and is a multiple of σ_k . This proves (5). Moreover, since Y is a pullback, it does not have any terms $d_{i,j}e^{-\lambda'_j s}g_j(t)$ at the negative end, where $j \neq -1$ modulo k . This proves (4). \square

Lemma 4.3.2. *Suppose (C1)–(C4) hold. After modifying $\{g_i(t)\}_{i \in \mathbb{Z} - \{0\}}$ subject to (E1) and (E2), there exists a basis $\{\sigma'_1, \dots, \sigma'_{k-1}, \sigma'_k = \sigma_k\}$ for $\ker(D_{v_0}^N)^*$ such that the following hold:*

(1) *The positive ends of σ'_i , $i = 1, \dots, k$, are of the form*

$$\sigma'_i(s, t) = e^{-\lambda_i s}f_i(t) + \sum_{j>i} c_{i,j}e^{-\lambda_j s}f_j(t).$$

(2) *The negative ends of σ'_i , $i = 1, \dots, k-1$, are of the form*

$$\sigma'_i(s, t) = d_{i,-k+i-1}e^{-\lambda'_{-k+i-1} s}g_{-k+i-1}(t) + \sum_{j<-k+i-1} d_{i,j}e^{-\lambda'_j s}g_j(t),$$

where $d_{i,-k+i-1}$ is nonzero.

(3) The negative end of σ'_k is of the form

$$\sigma'_k(s, t) = d_{k,-1} e^{-\lambda'_{-1}s} g_{-1}(t) \quad \text{modulo } g_{-k-1}, g_{-k-2}, \dots,$$

where $d_{k,-1}$ is nonzero.

(4) $\sigma'_k = Y$, up to a nonzero constant multiple.

Proof. Let $\{\sigma_1, \dots, \sigma_k\}$ be as in Lemma 4.3.1. We construct σ'_i by induction, starting with $\sigma'_k = \sigma_k$, which satisfies (3) and (4) by Lemma 4.3.1. Suppose σ'_{i+1} satisfies (1) and (2) and is of the form $\sigma_{i+1} + \sum_{j>i+1} k_j \sigma'_j$. We take σ'_i of the form $\sigma_i + \sum_{j>i} k_j \sigma'_j$ so that (1) and (2) are satisfied, with the possible exception of $d_{i,-k+i-1}$ being nonzero. If $\lambda'_{-k+i-2} < \lambda'_{-k+i-1}$, then $d_{i,-k+i-1} \neq 0$ is a consequence of Claim 4.2.4. On the other hand, if $\lambda'_{-k+i-2} = \lambda'_{-k+i-1}$, then $(d_{i,-k+i-1}, d_{i,-k+i-2}) \neq 0$ by Claim 4.2.4, and modifying g_{-k+i-1} and g_{-k+i-2} subject to (E1) and (E2) gives $d_{i,-k+i-1} \neq 0$. \square

5. THE DEFINITION OF $HC(\mathcal{D})$

5.1. The differential. Given an L -supersimple and L -monotone (M, α) and $J \in \mathcal{J}_*^{<L, \text{reg}}(\alpha)$, let $CC^L(M, \alpha, J)$ be the \mathbb{Q} -vector space generated by \mathcal{P}_α^L , the set of good Reeb orbits of R_α of action $< L$. Here an orbit γ is *bad* if it is an even multiple cover of a negative hyperbolic orbit; an orbit is *good* if it is not bad. Note that the generators do not depend on the choice of J .

If γ is an m -fold cover of a simple orbit, then we define the *multiplicity* $m(\gamma)$ of γ to be m . The differential ∂ of $CC^L(M, \alpha, J)$ is given by

$$\partial\gamma = \sum_{\gamma' \in \mathcal{P}_\alpha^L} \#(\mathcal{M}_J^{\text{ind}=1, \text{cyl}}(\gamma, \gamma')/\mathbb{R}) \frac{1}{m(\gamma')} \cdot \gamma',$$

where $\#$ refers to the signed count using the coherent orientation system from [BM]; see Section 9.1 for more details on orientations.

Remark 5.1.1. Although there is a denominator $\frac{1}{m(\gamma')}$ in the definition of ∂ , the coefficient of γ' is always an integer; the same holds for coefficients in chain maps. For example, when $(u, \mathbf{r}) \in \mathcal{M}_J^{\text{ind}=1, \text{cyl}}(\gamma, \gamma')$ and u is not multiply-covered (which is the case here), the contribution of all (u, \mathbf{r}) with the same image towards

$$\#(\mathcal{M}_J^{\text{ind}=1, \text{cyl}}(\gamma, \gamma')/\mathbb{R}) \frac{1}{m(\gamma')}$$

as we range over \mathbf{r} is $\pm m(\gamma)$.

Remark 5.1.2. In view of Remark 5.1.1, $CC^L(M, \alpha, J)$ can be defined over \mathbb{Z} . Similarly, the chain maps $\Phi_{X, \alpha, \overline{J}}$ can be defined over \mathbb{Z} . What is not defined over \mathbb{Z} are the chain homotopy maps K_\pm .

Theorem 5.1.3. *If J is generic, α is L -supersimple and L -monotone, and (α, J) is an L -supersimple pair, then the count $\#(\mathcal{M}_J^{\text{ind}=1, \text{cyl}}(\gamma, \gamma')/\mathbb{R})$ is finite and $\partial^2 = 0$.*

Proof. By Lemma 3.3.2, all the $\text{ind} = 1$ moduli spaces consist of simple curves. Hence $\mathcal{M}_J^{\text{ind}=1, \text{cyl}}(\gamma, \gamma')/\mathbb{R}$ is compact and transversely cut out. This implies the finiteness of $\#(\mathcal{M}_J^{\text{ind}=1, \text{cyl}}(\gamma, \gamma')/\mathbb{R})$.

For $\partial^2 = 0$, we consider $\text{ind} = 2$ moduli spaces $\mathcal{M} := \mathcal{M}_J^{\text{ind}=2, \text{cyl}}(\gamma, \gamma')$. By Lemma 3.3.2, either $u \in \mathcal{M}$ is simply-covered or double covers a simple u' with $\text{ind}(u') = 1$ and no branch points. In either case, Theorems 4.2.6 and 4.2.1 imply that \mathcal{M} is transversely cut out. By Lemma 3.3.2, every element u^∞ of $\partial\mathcal{M}$ is a two-level building $u_1 \cup u_2$, where u_2 is above u_1 , u_i , $i = 1, 2$, is simple, and $\text{ind}(u_i) = 1$. (By the monotonicity of α_\pm , every holomorphic plane in u^∞ must have $\text{ind} \geq 2$. Since $\text{ind} > 0$ for every nontrivial curve in $\mathbb{R} \times M$, there can be no planes. This implies that all the irreducible components of u^∞ are cylinders.) This implies $\partial^2 = 0$ modulo orientation considerations which are postponed until Section 9.1.

It remains to consider the case where u_2 is a curve from γ to γ'' and u_1 is a curve from γ'' to γ' , where γ'' is a bad Reeb orbit. Since γ'' is an even multiple cover of a negative hyperbolic orbit, the gluing occurs in canceling pairs. (Recall from [BM] that the orientation is reversed under a $\mathbb{Z}/2$ -deck transformation if γ'' is an even multiple of a negative hyperbolic orbit.) \square

5.2. Chain maps. Let (X^4, α) be a compact, connected, exact symplectic cobordism, where $d\alpha$ is symplectic, $\partial X = M_+ - M_-$, and $\alpha_\pm = \alpha|_{M_\pm}$ is a contact form on M_\pm . Let $(\widehat{X}, \widehat{\alpha})$ be the completion of (X, α) , obtained by attaching the symplectization ends $[0, \infty) \times M_+$ and $(-\infty, 0] \times M_-$, and let \overline{J} be an almost complex structure which tames $(\widehat{X}, \widehat{\alpha})$ and which restricts to α_\pm -tame almost complex structures J_\pm at the positive and negative ends. Also let $\mathcal{M}_{\overline{J}}(\gamma_+, \gamma_-)$ be the moduli space of \overline{J} -holomorphic maps in \widehat{X} from γ_+ to γ_- with markings, defined in a manner analogous to that of $(\mathbb{R} \times M, J)$.

Theorem 5.2.1. *If \overline{J} is generic, (M_+, α_+) and (M_-, α_-) are L -supersimple and L -monotone, and (α_+, J_+) and (α_-, J_-) are L -supersimple pairs, then $(X, \alpha, \overline{J})$ induces a chain map*

$$\Phi_{(X, \alpha, \overline{J})} : CC^L(M_+, \alpha_+, J_+) \rightarrow CC^L(M_-, \alpha_-, J_-).$$

Proof. The map $\Phi_{(X, \alpha, \overline{J})}$ is given by

$$\Phi_{(X, \alpha, \overline{J})}(\gamma_+) = \sum_{\gamma_- \in \mathcal{P}_{\alpha_-}^L} \#(\mathcal{M}_{\overline{J}}^{\text{ind}=0, \text{cyl}}(\gamma_+, \gamma_-)) \frac{1}{m(\gamma_-)} \cdot \gamma_-.$$

Let $(u, \mathbf{r}) \in \mathcal{M}_{\overline{J}}^{\text{ind}=0, \text{cyl}}(\gamma_+, \gamma_-)$. If u is not multiply-covered, then the contribution of all (u, \mathbf{r}) towards $\#(\mathcal{M}_{\overline{J}}^{\text{ind}=0, \text{cyl}}(\gamma_+, \gamma_-)) \frac{1}{m(\gamma_-)}$ as we range over \mathbf{r} is $\pm m(\gamma_-)$. On the other hand, if u is an a -fold cover of a simple curve, the contribution is $\pm m(\gamma_-)/a$.

Let us write $\mathcal{M}^i = \mathcal{M}_{\overline{J}}^{\text{ind}=i, \text{cyl}}(\gamma_+, \gamma_-)$. The proof is given in three steps.

Step 1: \mathcal{M}^0 is regular. Since simple curves in \mathcal{M}^0 are regular, it remains to prove that $v \in \mathcal{M}^0$ which is an unbranched m -fold cover of a simple u from γ to γ' is regular. By Theorem 4.2.6, we may assume that u is immersed. Let \dot{F}_1 and \dot{F}_0 be the domains of v and u and let $\pi : \dot{F}_1 \rightarrow \dot{F}_0$ be the covering map such that $v = u \circ \pi$.

Let N be a normal bundle of v as described in Section 4 and let τ be a trivialization of N so that either $\mu_\tau(\gamma) = \mu_\tau(\gamma') = 0$ or $\mu_\tau(\gamma) = \mu_\tau(\gamma') = 1$.

Suppose that $\mu_\tau(\gamma) = \mu_\tau(\gamma') = 1$. If m is even, then γ_- is an even multiple of γ' and is a bad orbit. Hence m is odd and $\#\Gamma_0(v) = 0$. Theorem 4.2.1 then implies that v is regular.

Next suppose that $\mu_\tau(\gamma) = \mu_\tau(\gamma') = 0$. By assumption u is regular. If v is not regular, then $0 \neq \zeta \in \ker(D_v^N)^*$ satisfies Equations (4.2.5) and (4.2.6), which implies the asymptotic condition

$$(5.2.1) \quad \text{wind}_\tau(\zeta, p_+) = \text{wind}_\tau(\zeta, p_-) = 0,$$

where p_+ (resp. p_-) is the puncture corresponding to the positive (resp. negative) end. We now consider the “trace map”

$$\pi_* : \ker(D_v^N)^* \rightarrow \ker(D_u^N)^*$$

which maps ζ to $\pi_*\zeta$, where $\pi_*\zeta(x) = \sum_{y \in \pi^{-1}(x)} \zeta(y)$. By Equation (5.2.1), $\pi_*\zeta \neq 0$, which is a contradiction. This proves the regularity of v .

Step 2: $\Phi = \Phi_{(X, \alpha, \bar{J})}$ is defined, i.e., \mathcal{M}^0 is finite. By the SFT compactness theorem [BEHWZ], if \mathcal{M}^0 is not finite, then there is a sequence $u_1, u_2, \dots \in \mathcal{M}^0$ that limits to an SFT building u^∞ . On the other hand, non-trivial simple curves inside $\mathbb{R} \times M_\pm$ are regular, so they have $\text{ind} > 0$, and simple curves inside \widehat{X} have $\text{ind} \geq 0$. Hence by Lemma 3.3.2 all the levels of u^∞ that map to $\mathbb{R} \times M_\pm$ have $\text{ind} > 0$ and the level of u^∞ that maps to the cobordism has $\text{ind} \geq 0$. Hence u^∞ is a 1-level building in \mathcal{M}^0 . This contradicts the regularity of \mathcal{M}^0 .

Step 3: $\partial \circ \Phi = \Phi \circ \partial$. First observe that \mathcal{M}^1 is regular since $\text{ind} = 1$ curves cannot be multiple covers by Lemma 3.3.2. We consider the boundary $\partial\mathcal{M}^1$ of \mathcal{M}^1 . If $u^\infty \in \partial\mathcal{M}^1$, then $u^\infty = v_0 \cup v_1$ or $v_{-1} \cup v_0$, where v_0 is a cylinder that maps to \widehat{X} and $v_i, i \neq 0$, is a cylinder that maps to $\mathbb{R} \times M_\pm$; the levels are arranged from bottom to top as i increases; $\text{ind}(v_i) = 1, i \neq 0$, and $\text{ind}(v_0) = 0$; and $v_i, i \neq 0$, is simple and v_0 may be multiply-covered. (By the monotonicity of α_\pm , every holomorphic plane in u^∞ must have $\text{ind} \geq 2$. Since $\text{ind} > 0$ for every nontrivial curve in $\mathbb{R} \times M_\pm$ and $\text{ind} \geq 0$ for every nontrivial curve in \widehat{X} , there can be no planes. This implies that each irreducible component of u^∞ is a cylinder.) Note that every $u \in \mathcal{M}^1$ is simple by Lemma 3.3.2. Suppose without loss of generality that $u^\infty = v_0 \cup v_1$. If v_0 is multiply-covered, then the positive end of v_0 cannot be a bad orbit, since otherwise the negative end γ_- of v_0 must also be a bad orbit. Hence v_0 is regular by Step 1. We conclude that all the elements of $\partial\mathcal{M}^1$ are two-level buildings, each of whose levels is regular. Hence $\partial \circ \Phi = \Phi \circ \partial$, once we observe that if we are gluing simple curves v_0 and v_1 along a bad orbit, then the gluing occurs in canceling pairs.

The orientation considerations are similar to those of Theorem 5.1.3. \square

5.3. Definition of $HC(\mathcal{D})$. We are now in a position to define the *cylindrical contact homology group* $HC(\mathcal{D})$. Given (M, ξ) , let

$$\mathcal{D} = (\alpha, \{L_i\}, \{\varphi_i\}, \{J_i\}, \{\bar{J}_i\})$$

be a tuple consisting of a nondegenerate α for (M, ξ) with no contractible Reeb orbits, sequences $\{L_i\}$ and $\{\varphi_i\}_{i=1}^\infty$ given by Corollary 2.0.4, a sequence $\{(\varphi_i\alpha, J_i)\}$ of L_i -supersimple pairs, and a sequence $\{\bar{J}_i\}$ of generic almost complex structures that are tamed by exact symplectic cobordisms on $\mathbb{R} \times M$ from $\varphi_i\alpha$ to $\varphi_{i+1}\alpha$ and that agree with J_i and J_{i+1} at the positive and negative ends.

The symplectic cobordisms together with \bar{J}_i induce chain maps

$$\Phi_i : CC^{L_i}(\varphi_i\alpha, J_i) \rightarrow CC^{L_{i+1}}(\varphi_{i+1}\alpha, J_{i+1}),$$

and we define $HC(\mathcal{D})$ as the direct limit of the induced maps $(\Phi_i)_*$ on homology.

6. THE EVALUATION MAP

The goal of this section is to introduce and discuss the properties of the evaluation map. Suppose that γ is a positive hyperbolic orbit of action $< L$; the situation of γ negative hyperbolic is similar.

Let γ be an $m(\gamma)$ -fold cover of a simple orbit γ_0 , let $\gamma_0 \times D_{\delta_0/3}^2$ be a neighborhood of γ_0 , where δ_0 is as in Section 3.2, and let $(\mathbb{R}/\mathbb{Z}) \times D_{\delta_0/3}^2 \rightarrow \gamma_0 \times D_{\delta_0/3}^2$ be its $m(\gamma)$ -fold cover with coordinates $(t, z = x + iy)$ such that $\{z = 0\}$ corresponds to γ . Also let $\mathbb{R} \times (\mathbb{R}/\mathbb{Z}) \times D_{\delta_0/3}^2$ be the cylinder over $(\mathbb{R}/\mathbb{Z}) \times D^2$ with coordinates $(s, t, z = x + iy)$.

Let

$$(6.0.1) \quad \cdots \leq \lambda_{-2} \leq \lambda_{-1} < 0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

be the eigenvalues of $A = -j_0 \frac{\partial}{\partial t} - S$ with $S = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}$ and let $f_i(t)$ be an eigenfunction corresponding to λ_i with L^2 -norm 1 so that $\{f_i(t)\}_{i \neq 0}$ is an orthonormal basis for $L^2(S^1, \mathbb{R}^2)$.

Fact 6.0.1. *If $J \in \mathcal{J}_*^{<L, \text{reg}}$ and*

$$u : (-\infty, R] \times S^1 \rightarrow \mathbb{R} \times M$$

is a J -holomorphic half-cylinder which is negatively asymptotic to γ for some R , then there exists $R' \ll 0$ such that $u(s, t)$, after reparametrization of the domain and lifting to the $m(\gamma)$ -fold cover $\mathbb{R} \times (\mathbb{R}/\mathbb{Z}) \times D_{\delta_0/3}^2$, can be written on $\{s \leq R'\}$ as:

$$(6.0.2) \quad \tilde{u}(s, t) = \left(s, t, \sum_{i=1}^{\infty} c_i e^{\lambda_i s} f_i(t) \right) \in \mathbb{R} \times (\mathbb{R}/\mathbb{Z}) \times D_{\delta_0/3}^2.$$

By abuse of notation we will usually not distinguish between u and \tilde{u} . We refer to Equation (6.0.2) as the “Fourier series” for u . The real constants c_i will be referred to as the “Fourier coefficients”. We also define the *order k Fourier polynomial* $P_k(u)$ of u as:

$$P_k(u) = \sum_{i=1}^k c_i e^{\lambda_i s} f_i(t).$$

Let $J \in \mathcal{J}_*^{<L, \text{reg}}$ and let $\mathcal{A}_{\alpha_+}(\gamma_+), \mathcal{A}_{\alpha_+}(\gamma'_+) < L$. We define the *order k evaluation map at the negative end*

$$\begin{aligned} ev_-^k(\gamma_+, \gamma'_+, J) : \mathcal{M}_J^{\text{cyl}}(\gamma_+, \gamma'_+) &\rightarrow \mathbb{R}^k \\ (u, r_+, r_-) &\mapsto (c_1, \dots, c_k), \end{aligned}$$

where u agrees with a J -holomorphic half-cylinder $(-\infty, R] \times S^1 \rightarrow \mathbb{R} \times M$ which is negatively asymptotic to γ'_+ for some R and has Fourier coefficients c_1, \dots, c_k . Here we parametrize \mathbb{R}/\mathbb{Z} such that the asymptotic marker r_- at the negative end corresponds to $t = 0$.

Fact 6.0.2. *The map $ev_-^k(\gamma_+, \gamma'_+, J)$ is smooth.*

The moduli space $\mathcal{M}_J^{\text{cyl}}(\gamma_+, \gamma'_+)$ admits the usual \mathbb{R} -translation which corresponds to the \mathbb{R}^+ -action on \mathbb{R}^k given by:

$$(c_1, \dots, c_k) \mapsto (c_1 e^{\lambda_1 s}, \dots, c_k e^{\lambda_k s}).$$

Provided there is no u with $(c_1, \dots, c_k) = 0$, $ev_-^k(\gamma_+, \gamma'_+, J)$ descends to the quotient

$$(6.0.3) \quad \tilde{ev}_-^k(\gamma_+, \gamma'_+, J) : \mathcal{M}_J^{\text{cyl}}(\gamma_+, \gamma'_+)/\mathbb{R} \rightarrow \mathbb{R}^k/\mathbb{R}^+ \simeq S^{k-1}.$$

Remark 6.0.3. We will make more precise the identification $\mathbb{R}^k/\mathbb{R}^+ \simeq S^{k-1}$: Each path

$$\mathbf{c} : \mathbb{R} \rightarrow \mathbb{R}^k, \quad s \mapsto (c_1 e^{\lambda_1 s}, \dots, c_k e^{\lambda_k s})$$

is transverse to the spheres $S_r^{k-1} = \{|x| \in \mathbb{R}^k \mid |x| = r\}$ since $\langle \mathbf{c}'(s), \mathbf{c}(s) \rangle > 0$ for all $s \in \mathbb{R}$. We will take the representative of \mathbf{c} to be its intersection with S_r^{k-1} , where r is taken to be 1 unless specified otherwise.

The maps

$$\begin{aligned} ev_+^k(\gamma'_-, \gamma_-, J) : \mathcal{M}_J^{\text{cyl}}(\gamma'_-, \gamma_-) &\rightarrow \mathbb{R}^k, \\ \tilde{ev}_+^k(\gamma'_-, \gamma_-, J) : \mathcal{M}_J^{\text{cyl}}(\gamma'_-, \gamma_-)/\mathbb{R} &\rightarrow S^{k-1} \end{aligned}$$

are defined similarly.

The main results of this subsection concern the transversality properties of the order k evaluation map, which generalizes [HT2, Proposition 2.2] in a special case. Recall the superscript δ_0 from Section 3.3. Let

$$\pi_{\mathbb{R}} : \mathcal{M}(\gamma_+, \gamma'_+) \rightarrow \mathcal{M}(\gamma_+, \gamma'_+)/\mathbb{R}$$

be the quotient map by translations in the s -direction.

Theorem 6.0.4. *Given $J \in \mathcal{J}_*^{<L, \text{reg}, \delta_0}$, a compact domain $K \subset \mathcal{M}_J^{\text{cyl}, s}(\gamma_+, \gamma'_+)/\mathbb{R}$, and a submanifold $Z \subset S^{k-1}$, there exist an arbitrarily close $J' \in \mathcal{J}_*^{<L, \text{reg}, \delta'_0}$ with $\delta'_0 < \delta_0$ and a compact domain $K' \subset \mathcal{M}_{J'}^{\text{cyl}, s}(\gamma_+, \gamma'_+)/\mathbb{R}$ such that:*

- (1) $\pi_{\mathbb{R}}^{-1}(K')$ contains all the elements of $\mathcal{M}_{J'}^{\text{cyl}, s}(\gamma_+, \gamma'_+)$ that are sufficiently close to $\pi_{\mathbb{R}}^{-1}(K)$; and
- (2) the restriction of the evaluation map $ev_-^k(\gamma_+, \gamma'_+, J')$ to $\pi_{\mathbb{R}}^{-1}(K')$ descends to $\tilde{ev}_-^k(\gamma_+, \gamma'_+, J')$ and is transverse to Z .

Remark 6.0.5. The proof is modeled on but is substantially easier than that of [HT2, Proposition 2.2]. This is due to the fact that the J -holomorphic equation is linear near each $\mathbb{R} \times \gamma$. This allows us to dispense with the quadratic estimates.

Proof. Let \mathcal{J}_J be the subset of $\mathcal{J}_*^{<L, \text{reg}, \delta_0}$ consisting of J' that are C^1 -close to J . Also let $\pi : \mathcal{M} \rightarrow \mathcal{J}_J$ be a bundle with fiber $\pi^{-1}(J') = \mathcal{M}_{J'}^{\text{cyl}}(\gamma_+, \gamma'_+)$ and let $\mathcal{M}^s \subset \mathcal{M}$ be the subset of simple curves. We will show that the map

$$Ev_-^k(\gamma_+, \gamma'_+) : \mathcal{M}^s \rightarrow \mathbb{R}^k$$

$$(u, J') \mapsto ev_-^k(\gamma_+, \gamma'_+, J')(u).$$

is transverse to the preimage of Z in \mathbb{R}^k at all points of K . Theorem 6.0.4 then follows from Sard's theorem.

Let $u \in K$ and $(\xi, Y) \in T_{(u, J)}\mathcal{M}^s$. With respect to the usual coordinates (t, x, y) on $(\mathbb{R}/\mathbb{Z}) \times D_{\delta_0}^2$, the almost complex structure J maps $\frac{\partial}{\partial s} \mapsto \frac{\partial}{\partial t} + X_H$ and $\frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial y}$ on $D_{\delta_0/3}^2$. We assume that Y corresponds to the path $J + \tau Y$, $\tau \in [-\varepsilon, \varepsilon]$, which maps $\frac{\partial}{\partial s} \mapsto \frac{\partial}{\partial t} + X_H$ and leaves $TD_{\delta_0/3}^2$ invariant. We write $j_0 + \tau Y_0$ for the restriction of $J + \tau Y$ to $TD_{\delta_0/3}^2$.

If $\bar{\partial}_{J+\tau Y}(u + \tau \xi) = 0$ for $\tau \in [-\varepsilon, \varepsilon]$, then we claim that (ξ, Y) satisfies

$$(6.0.4) \quad \frac{\partial \xi}{\partial s} + j_0 \frac{\partial \xi}{\partial t} - j_0 \left(X_H(\xi) + Y_0(s, t, u(s, t)) \left(\frac{\partial u}{\partial s} \right) \right) = 0.$$

Since we are only concerned with the negative end of u , we assume without loss of generality that u is graphical over $(-\infty, 0] \times (\mathbb{R}/\mathbb{Z})$, has image in $(-\infty, 0] \times (\mathbb{R}/\mathbb{Z}) \times D_{\delta_0/3}^2$, and is written as $(s, t, u(s, t))$. Similarly we write ξ as $(s, t, \xi(s, t))$.

The almost complex structure $J + \tau Y$ maps

$$\left(1, 0, \frac{\partial(u + \tau \xi)}{\partial s} \right) \mapsto \left(0, 1, j_0 \frac{\partial(u + \tau \xi)}{\partial s} + X_H(u + \tau \xi) + \tau Y_0 \left(\frac{\partial(u + \tau \xi)}{\partial s} \right) \right),$$

which must equal $(0, 1, \frac{\partial(u + \tau \xi)}{\partial t})$. Equating the third coordinates, using $j_0 \frac{\partial u}{\partial s} + X_H(u) = \frac{\partial u}{\partial t}$, and differentiating with respect to τ , we obtain Equation (6.0.4).

Case 1. First suppose that u does not intersect $\mathbb{R} \times \gamma'_+$.

We solve for (ξ, Y) in Equation (6.0.4), i.e., on the negative end, where

$$(6.0.5) \quad j_0 Y_0(s, t, u(s, t)) \left(\frac{\partial u}{\partial s} \right) = \mu(s) f_i(t)$$

and $\xi(s, t) = \rho(s)f_i(t)$ for $i \geq 1$, μ and ρ are smooth in s , $\mu(s) \geq 0$ is a function with support on $[R', R]$ for some $R', R \in \mathbb{R}$ and total integral 1, and $\rho(s) = 0$ for $s > R$.

Equation (6.0.4) then becomes

$$\begin{aligned}\rho'(s)f_i(t) - \rho(s)\lambda_i f_i(t) - \mu(s)f_i(t) &= 0, \\ \rho'(s) - \lambda_i \rho(s) &= \mu(s).\end{aligned}$$

We then pick the solution

$$(6.0.6) \quad \rho(s) = e^{\lambda_i s} \int_s^R e^{-\lambda_i \sigma} \mu(\sigma) d\sigma.$$

Then $\xi(s, t) = \rho(s)f_i(t)$ is equal to 0 on $s = R$ and can be written as

$$(6.0.7) \quad \xi(s, t) = ce^{\lambda_i s} f_i(t), \quad c \geq e^{-\lambda_i R} / \lambda_i$$

on $s \leq R'$. (Note that R is a large negative number.)

We then consider the extension of (ξ, Y) . We extend ξ to all of the domain of u by setting $\xi = 0$. Let u_T be the curve obtained from u by translating in the s -direction by T units. By the following claim, for sufficiently negative R' and R , Y can be extended from the annulus $\text{Im } u|_{R' \leq s \leq R}$ to all of $\mathbb{R} \times M$ so that Y is s -invariant and has support on

$$\coprod_{T \in \mathbb{R}} \text{Im } u_T|_{R' \leq s \leq R} \subset \mathbb{R} \times (\mathbb{R}/\mathbb{Z}) \times (D_{\delta_0/3}^2 - D_{\delta'_0/3}^2)$$

for some $0 < \delta'_0 < \delta_0$.

Claim 6.0.6. For $s_0 \ll 0$, $u|_{s=s_0}(t) = \sum_{i=1}^{\infty} c_i e^{\lambda_i s_0} f_i(t)$ is an embedding.

Proof of Claim 6.0.6. The reason the claim is not trivial is that γ'_+ may be an m -fold cover of a simple orbit γ_0 . Since u is simple by assumption, the negative end of u is not multiply-covered. Hence there exists an integer $j > 0$ such that:

- for each $c_i \neq 0$ with $i < j$, $f_i(t)$ is an a -fold cover of an asymptotic eigenfunction for $\gamma_0^{m/a}$, where $a > 1$, and
- $c_j \neq 0$ and $f_j(t)$ is a b -fold cover of an asymptotic eigenfunction for $\gamma_0^{m/b}$, where $(a, b) = 1$.

By an explicit calculation of asymptotic eigenfunctions (i.e., Equations (4.1.1)–(4.1.4)), it follows that $u|_{s=s_0}$ is an embedding for $s_0 \ll 0$. \square

Since u does not intersect $\mathbb{R} \times \gamma'_+$, the support of Y intersects u only near the negative end of u . Hence the pair (ξ, Y) , described above and satisfying Equation (6.0.7), is indeed an element of $T_{(u, J)}\mathcal{M}^s$ and the theorem holds in this case.

Case 2. Suppose that $u : \dot{F} \rightarrow \mathbb{R} \times M$ nontrivially intersects $\mathbb{R} \times \gamma'_+$. Let Y be as in Case 1 and let u^*Y be the pullback of Y to $u^*\text{End}(T(\mathbb{R} \times M))$. Then u^*Y can be written as $Y' + Y''$, where Y' is supported on the negative end of \dot{F} and Y'' is supported on a neighborhood of $Z \subset \dot{F}$. Here Z is the preimage of $u(\dot{F}) \cap (\mathbb{R} \times \gamma'_+)$.

Let ξ' be the solution to $\bar{\partial}_{J+\tau Y'}(u+\tau\xi') = 0$ up to first order in τ , as constructed in Case 1 (the notation in Case 1 is (ξ, Y)) and whose negative end satisfies Equation (6.0.7).

Let ξ'' be the solution to $\bar{\partial}_{J+\tau Y''}(u+\tau\xi'') = 0$ up to first order in τ , which is L^2 -orthogonal to the kernel of the linearized $\bar{\partial}$ -operator $D_{u,J}$. We now estimate that the solution ξ'' corresponding to Y'' is much smaller than ξ' along $s = R'$ if $R' \ll R \ll 0$. We will use Morrey space norms which are defined in Section 8.3 and ideas that are used in Section 8.7. The constant $c > 0$ may change from line to line. We first observe that $\xi'' = D_{u,J}^{-1}(\zeta)$, where $\|\zeta\| \leq c\|Y\|$. Since $D_{u,J}^{-1}$ is bounded, $\|\xi''\|_* \leq c\|Y\|$. Then, by Lemma 8.3.1, $|\xi''|_{C^0} \leq c\|Y\|$. Hence $|\xi''|_{s=R}$ has the same order of magnitude as $|\xi'|_{C^0}$. However, since ξ'' decays exponentially as $s \rightarrow -\infty$,

$$(6.0.8) \quad |\xi''|_{s=R'} \leq ce^{-\lambda(R-R')} |\xi''|_{s=R},$$

where $\lambda = \min(\lambda_1, |\lambda_{-1}|)$. This implies that $|\xi''|_{s=R'} \ll |\xi'|_{s=R'}$.

Since $\xi' + \xi''$ is a solution corresponding to $Y' + Y''$, the theorem follows. \square

Let us abbreviate $\mathcal{M} = \mathcal{M}_J^{\text{ind}=k, \text{cyl}}(\gamma_+, \gamma'_+)$ and $\tilde{e}v_-^k = \tilde{e}v_-^k(\gamma_+, \gamma'_+, J)$, where $k \geq 2$. We also use the superscript “sing” to denote the subset of \mathcal{M} consisting of curves with singularities.

Theorem 6.0.7. *For a generic $J \in \mathcal{J}_*^{<L, \text{reg}}$, $\tilde{e}v_-^k : \mathcal{M}/\mathbb{R} \rightarrow S^{k-1}$ satisfies the following:*

- (1) *The restriction of $\tilde{e}v_-^k$ to $(\mathcal{M} - \mathcal{M}^{\text{sing}})/\mathbb{R}$ is an immersion.*
- (2) *$\tilde{e}v_-^k(\mathcal{M}^{\text{sing}}/\mathbb{R})$ has codimension at least 2 and is disjoint from $(0, \dots, 0, \pm 1)$.*
- (3) *In particular, $(0, \dots, 0, \pm 1)$ are regular values of $\tilde{e}v_-^k$.*

Proof. (1) Let $u \in \mathcal{M} - \mathcal{M}^{\text{sing}}$. Since u is immersed and $\text{ind}(u) = k \geq 2$, u is regular by Theorem 4.2.1. By the argument of Lemma 1.0.5, there exists a basis $\{e_1, \dots, e_k\}$ for $\ker D_u$, such that the negative ends of e_i are of the form

$$e_i(s, t) = e^{\lambda_i s} f_i(t) \quad \text{modulo } f_{k+1}, f_{k+2}, \dots$$

This implies the surjectivity of $(\tilde{e}v_-^k)_* : T_{[u]}(\mathcal{M}/\mathbb{R}) \rightarrow T_{\tilde{e}v_-^k([u])} S^{k-1}$.

(2) $\mathcal{M}^{s, \text{sing}} \subset \mathcal{M}^s$ has (real) codimension at least two by Theorem 4.2.6. Hence $\tilde{e}v_-^k(\mathcal{M}^{s, \text{sing}}/\mathbb{R})$ has codimension at least two and is disjoint from $\{(0, \dots, 0, \pm 1)\}$ by Theorem 6.0.4. (We will be a little sloppy: The issue here is to find a fixed radius δ'_0 which works for all of $\overline{\mathcal{M}}/\mathbb{R}$. Strictly speaking, Theorem 6.0.4 holds for a large compact subset $K \subset \mathcal{M}^{s, \text{sing}}/\mathbb{R}$. We then apply Theorem 6.0.4 to the strata of the boundary $\overline{\mathcal{M}}/\mathbb{R}$ to obtain Theorem 6.0.8 below. This in turn allows us to use $\mathcal{M}^{s, \text{sing}}/\mathbb{R}$ instead of K .)

It remains to consider the multiply-covered curves in \mathcal{M} . Writing $\gamma_+ = \gamma_{+,0}^b$ and $\gamma'_+ = (\gamma'_{+,0})^b$, let $\mathcal{S}' = \mathcal{M}_J^{\text{ind}=a, \text{cyl}}(\gamma_{+,0}, \gamma'_{+,0})$, where $ab = k$, $b > 1$, and let $\mathcal{S} \subset \mathcal{M}$ be the set of b -fold covers of curves in \mathcal{S}' . By induction, suppose that

$$\tilde{e}v_-^a(\gamma_{+,0}, \gamma'_{+,0}, J) : \mathcal{S}'/\mathbb{R} \rightarrow S^{a-1}$$

satisfies (2) with k replaced by a . (As the initial step of the induction, observe that if $k = 2$ then \mathcal{M} has no singular curves by Theorem 4.2.6 and (2) holds.) Then $\tilde{e}v_-^a$ can be “lifted” to $\tilde{e}v_-^k : \mathcal{S}/\mathbb{R} \rightarrow S^{ab-1}$ as follows: Given $v \in \mathcal{S}$, suppose it is the b -fold cover of $u \in \mathcal{S}'$. Then $\tilde{e}v_-^k(v) = i \circ \tilde{e}v_-^a(u)$, where the inclusion $i : S^{a-1} \rightarrow S^{ab-1}$ is induced by

$$(6.0.9) \quad \mathbb{R}^a \rightarrow \mathbb{R}^{ab}, \quad (x_1, \dots, x_a) \mapsto (x_1, 0, \dots, 0, x_2, 0, \dots, 0, x_3, 0, \dots)$$

(the zeros are inserted in the same positions for all (x_1, \dots, x_a)). While it is possible for $\tilde{e}v_-^k(\mathcal{S})$ to pass through $(0, \dots, 0, \pm 1) \in S^{k-1}$, $\tilde{e}v_-^k(\mathcal{S}^{\text{sing}})$ will not pass through $(0, \dots, 0, \pm 1)$ by the induction hypothesis. This proves (2).

(1) and (2) then imply (3). \square

Finally we consider the extension $\overline{e}v_-^k : \overline{\mathcal{M}}/\mathbb{R} \rightarrow S^{k-1}$ of $\tilde{e}v_-^k$ to the compactification of \mathcal{M}/\mathbb{R} .

Theorem 6.0.8. *For a generic $J \in \mathcal{J}_*^{<L, \text{reg}}$, $\overline{e}v_-^k : \overline{\mathcal{M}}/\mathbb{R} \rightarrow S^{k-1}$ satisfies the following:*

- (1) $\overline{e}v_-^k(\partial(\mathcal{M}/\mathbb{R}))$ is disjoint from $(0, \dots, 0, \pm 1)$.
- (2) If \mathcal{S} is a (nonempty) stratum of $\partial(\mathcal{M}/\mathbb{R})$ consisting of l -level buildings $u_1 \cup \dots \cup u_l$ with $\text{ind}(u_i) = a_i$, then $\dim(\overline{e}v_-^k(\mathcal{S})) = a_1 - 1$ and $\mathcal{S}^{\text{sing}}$ has codimension at least 2 in \mathcal{S} .

As usual, u_1 is the lowest level and u_l is the highest level.

Proof. We apply Theorem 6.0.7 to the $\text{ind} = a_1$ moduli space \mathcal{N} containing u_1 as given in the statement of (2) and the evaluation map $\tilde{e}v_-^{a_1} : \mathcal{N}/\mathbb{R} \rightarrow S^{a_1-1}$. We then extend the map to $\tilde{e}v_-^k : \mathcal{N}/\mathbb{R} \rightarrow S^{k-1}$ to obtain (1) and (2). The details are left to the reader. \square

7. CHAIN HOMOTOPY

7.1. Chain homotopy. Let X^4 be a compact connected 4-manifold such that $\partial X = M_+ - M_-$, let $\alpha|_{M_\pm}$ be a contact form on M_\pm , and let $\{\alpha^\tau\}_{0 \leq \tau \leq 1}$ be a 1-parameter family of 1-forms on X such that $d\alpha^\tau$ is symplectic and $\alpha^\tau|_{M_\pm} = \alpha_\pm$, for all $\tau \in [0, 1]$. Also let $\{(\hat{X}^\tau = \hat{X}, \hat{\alpha}^\tau)\}_{0 \leq \tau \leq 1}$ be a 1-parameter family of completions of $\{(X^\tau, \alpha^\tau)\}_{0 \leq \tau \leq 1}$ and let \overline{J}^τ be an almost complex structure which tames $(\hat{X}^\tau, \hat{\alpha}^\tau)$ and which restricts to α_\pm -tame almost complex structures J_\pm near the positive and negative ends. *From now on we further require that all contact forms are L -noncontractible.*

Theorem 7.1.1. *Suppose that (M_-, α_-) and (M_+, α_+) are L -supersimple and L -noncontractible, $\{\overline{J}^\tau\}_{0 \leq \tau \leq 1}$ is generic as a family, and (α_+, J_+) and (α_-, J_-) are L -supersimple pairs. Then the chain maps $\Phi_{(X^1, \alpha^1, \overline{J}^1)}$ and $\Phi_{(X^0, \alpha^0, \overline{J}^0)}$ defined in Theorem 5.2.1 induce the same map on homology.*

To prove this theorem, we show there exist linear maps

$$K_\pm : CC^L(M_+, \alpha_+, J_+) \rightarrow CC^L(M_-, \alpha_-, J_-)$$

such that for any $\gamma_+ \in \mathcal{P}_{\alpha_+}^L$, one has

$$(7.1.1) \quad \Phi_{(X^1, \alpha^1, \bar{J}^1)}(\gamma_+) - \Phi_{(X^0, \alpha^0, \bar{J}^0)}(\gamma_+) = K_+ \partial_+(\gamma_+) + \partial_- K_-(\gamma_+).$$

We consider the 1-dimensional moduli space

$$\mathcal{M}^0 = \coprod_{0 \leq \tau \leq 1} \mathcal{M}_{\bar{J}^\tau}^{\text{ind}=0, \text{cyl}}(\gamma_+, \gamma_-)$$

and explain how each component of $\partial \mathcal{M}^0$ contributes to Equation (7.1.1).

We first establish some notation. An SFT building u^∞ will be written as:

$$u^\infty = v_{-b} \cup \cdots \cup v_{-1} \cup v_0 \cup v_1 \cup \cdots \cup v_a$$

where the levels are arranged from bottom to top as we go from left to right; v_j , $j < 0$, maps to $(\mathbb{R} \times M, J_-)$; v_0 maps to \widehat{X}^τ for some τ ; v_j , $j > 0$, maps to $(\mathbb{R} \times M, J_+)$; the v_j 's are all holomorphic cylinders; and the pregluing of all the v_j yields a cylinder of $\text{ind} = 0$. The only possible level with negative Fredholm index is v_0 .

Simple \mathcal{M}^0 case. Suppose first that no curve of \mathcal{M}^0 is a multiple cover.

The boundary $\partial \mathcal{M}^0 = \overline{\mathcal{M}^0} - \mathcal{M}^0$ admits a decomposition

$$\partial_1 \mathcal{M}^0 \coprod \partial_2 \mathcal{M}^0 \coprod \partial_3 \mathcal{M}^0,$$

each of which will be discussed below.

Type 1. $\partial_1 \mathcal{M}^0$ corresponds to the case when $\tau = 0$ or 1, i.e.,

$$\partial_1 \mathcal{M}^0 = \mathcal{M}_{\bar{J}^0}^{\text{ind}=0, \text{cyl}}(\gamma_+, \gamma_-) \coprod \mathcal{M}_{\bar{J}^1}^{\text{ind}=0, \text{cyl}}(\gamma_+, \gamma_-).$$

They contribute to

$$m(\gamma_-) \cdot \left\langle \Phi_{(X^1, \alpha^1, \bar{J}^1)}(\gamma_+) - \Phi_{(X^0, \alpha^0, \bar{J}^0)}(\gamma_+), \gamma_- \right\rangle.$$

Type 2. $\partial_2 \mathcal{M}^0$ corresponds to the case when curves in \mathcal{M}^0 converge to two-level holomorphic buildings: $v_0 \cup v_1$ or $v_{-1} \cup v_0$, where v_{-1} , v_0 and v_1 are all immersions and v_0 is a k -fold cover of a simple holomorphic cylinder u_0 of index -1 . We denote the sets of these two types of two-level buildings by $\partial_2^+ \mathcal{M}^0$ and $\partial_2^- \mathcal{M}^0$ respectively, so that $\partial_2 \mathcal{M}^0 = \partial_2^+ \mathcal{M}^0 \coprod \partial_2^- \mathcal{M}^0$.

Type 3. $\partial_3 \mathcal{M}^0$ consists of higher codimension strata.

Case I. $v_{-1} \cup v_0$ or $v_0 \cup v_1$, where $v_{\pm 1}$ is a singular curve in $\mathbb{R} \times M_{\pm}$.

Case II. $v_{-l} \cup \cdots \cup v_0$ or $v_0 \cup \cdots \cup v_l$, where $l > 1$.

Case III. $v_{-l} \cup \cdots \cup v_0 \cup \cdots \cup v_{l'}$, where $l, l' > 0$.

Lemma 7.1.2. $\partial_3 \mathcal{M}^0 = \emptyset$.

Proof. Cases I and II cannot occur by Proposition 8.8.1. Case III is eliminated in Section 8.9. \square

Now we go back to analyze the contributions from the Type 2 boundary. We will focus on $\partial_2^+ \mathcal{M}^0$ and show that it corresponds to the term $\langle K_+ \partial_+(\gamma_+), \gamma_- \rangle$. $\partial_2^- \mathcal{M}^0$ can be dealt with similarly and corresponds to the term $\langle \partial_- K_-(\gamma_+), \gamma_- \rangle$.

We first consider the moduli spaces of the form $\mathcal{M}_{\overline{J}^{\tau_l}}^{\text{ind}=-k, \text{cyl}}(\gamma'_+, \gamma_-)$ for $k \geq 1$. There are only finitely many $\tau_l \in [0, 1]$ such that $\mathcal{M}_{\overline{J}^{\tau_l}}^{\text{ind}=-k, \text{cyl}}(\gamma'_+, \gamma_-)$ is non-empty since $\{\overline{J}^{\tau}\}$ is generic; and, for each such τ_l , if $(v, \mathbf{r}) \in \mathcal{M}_{\overline{J}^{\tau_l}}^{\text{ind}=-k, \text{cyl}}(\gamma'_+, \gamma_-)$, then v is a k -fold cover of a simple curve u , where $(u, \mathbf{r}') \in \mathcal{M}_{\overline{J}^{\tau_l}}^{\text{ind}=-1, \text{cyl}}((\gamma'_+)^{1/k}, \gamma_-^{1/k})$, γ'_+ is the k -fold cover of $(\gamma'_+)^{1/k}$, and γ_- is the k -fold cover of $\gamma_-^{1/k}$.

Remark 7.1.3. For each \overline{J}^{τ_l} -holomorphic cylinder from $(\gamma'_+)^{1/k}$ to $\gamma_-^{1/k}$ of index -1 , without markers but after quotienting by automorphisms, there are $m((\gamma'_+)^{1/k}) \cdot m(\gamma_-^{1/k})$ elements of $\mathcal{M}_{\overline{J}^{\tau_l}}^{\text{ind}=-1, \text{cyl}}((\gamma'_+)^{1/k}, \gamma_-^{1/k})$ and $k \cdot m((\gamma'_+)^{1/k}) \cdot m(\gamma_-^{1/k})$ elements of $\mathcal{M}_{\overline{J}^{\tau_l}}^{\text{ind}=-k, \text{cyl}}(\gamma'_+, \gamma_-)$.

Lemma 7.1.4. *For each integer $k \geq 1$ and $R \gg 0$, there exists a $(k-1)$ -dimensional vector bundle*

$$\mathcal{O}_{+,k} \rightarrow [R, \infty) \times \coprod_{\gamma'_+ \in \mathcal{P}_{\alpha_+}^L} \left(\coprod_{0 \leq \tau \leq 1} \mathcal{M}_{\overline{J}^{\tau}}^{\text{ind}=-k, \text{cyl}}(\gamma'_+, \gamma_-) \times \mathcal{M}_{J_+}^{\text{ind}=k, \text{cyl}}(\gamma_+, \gamma'_+)/\mathbb{R} \right),$$

called the obstruction bundle, and an obstruction section $\mathfrak{s}_{+,k}$ of $\mathcal{O}_{+,k}$, for which there exists a neighborhood $\mathcal{N} \subset \overline{\mathcal{M}}^0$ of $\partial_2^+ \mathcal{M}^0$ such that

$$\mathcal{N} - \partial_2^+ \mathcal{M}^0 = \coprod_{k \geq 1} (\mathfrak{s}_{+,k})^{-1}(0).$$

Remark 7.1.5. When $k = 1$, we have the usual gluing of $v_0 \cup v_1$, where $\text{ind}(v_0) = -1$, $\text{ind}(v_1) = 1$, v_0 maps to some \widehat{X}^{τ_0} , v_1 maps to $\mathbb{R} \times M$, and v_0, v_1 are simple.

The explicit definitions of $\mathcal{O}_{+,k}$ and $\mathfrak{s}_{+,k}$ are given in Section 8.5 — there they are written as \mathcal{O} and \mathfrak{s} with the understanding that k and (v_0, \mathbf{r}_0) are fixed. The proof of Lemma 7.1.4 follows from Theorem 8.6.1.

Recall that we are viewing $((v_0, \mathbf{r}_0), (v_1, \mathbf{r}_1)) \sim ((v'_0, \mathbf{r}'_0), (v'_1, \mathbf{r}'_1))$ if there exist automorphisms π_1 and π_2 of the domains $\dot{F}_1 = \mathbb{R} \times S^1$ and $\dot{F}_2 = \mathbb{R} \times S^1$ such that $v_1 = v'_1 \circ \pi_1$, $v_2 = v'_2 \circ \pi_2$, and π_1 and π_2 take positive (resp. negative) punctures to positive (resp. negative) punctures but are not necessarily marker-preserving.

Source of \mathbb{Q} -coefficients. It is tempting to further identify

$$(7.1.2) \quad ((v_0, \mathbf{r}_0), (v_1, \mathbf{r}_1)) \sim' ((v'_0, \mathbf{r}'_0), (v'_1, \mathbf{r}'_1))$$

if

$$(7.1.3) \quad (r_{0+} - r_{0-}) + (r_{1+} - r_{1-}) = (r'_{0+} - r'_{0-}) + (r'_{1+} - r'_{1-}).$$

Here the asymptotic markers are viewed as elements of $S^1 \in \mathbb{R}/\mathbb{Z}$. In the $k = 1$ case, the pairs identified by \sim' represent the same boundary point of \mathcal{M}^0 . However, for $k > 1$, unless $((v_0, \mathbf{r}_0), (v_1, \mathbf{r}_1)) = ((v'_0, \mathbf{r}'_0), (v'_1, \mathbf{r}'_1))$, the upper level

of the 2-level buildings identified by \sim' do not continue to the same component of $(\overline{ev}_-^k)^{-1}(\nu)$; cf. the proof of Theorem 7.1.1. This is the source of “branching”, which in turn forces us to use \mathbb{Q} -coefficients.

Define $n_{+, \tau_l}^k(\gamma_+, \gamma'_+; \gamma_-)$ as the signed count

$$\# \left(\mathfrak{s}_{+, k}^{-1}(0) \cap \left(\{T\} \times \mathcal{M}_{\overline{J}_l}^{\text{ind}=-k, \text{cyl}}(\gamma'_+, \gamma_-) \times \mathcal{M}_{J_+}^{\text{ind}=k, \text{cyl}}(\gamma_+, \gamma'_+)/\mathbb{R} \right) \right) \frac{1}{m(\gamma'_+)}$$

for generic $T \in [R, \infty)$. Then, in view of Lemma 7.1.2,

(7.1.4)

$$\begin{aligned} \left\langle \left(\Phi_{(X^1, \alpha^1, \overline{J}^1)} - \Phi_{(X^0, \alpha^0, \overline{J}^0)} \right) (\gamma_+, \gamma_-) \right\rangle &= \sum_{k=1}^{\infty} \sum_{\gamma'_+ \in \mathcal{P}_{\alpha_+}^L} \sum_l n_{+, \tau_l}^k(\gamma_+, \gamma'_+; \gamma_-) \frac{1}{m(\gamma_-)} \\ &\quad + \text{ terms coming from } \partial_2^- \mathcal{M}^0. \end{aligned}$$

Recall the evaluation map

$$\overline{ev}_-^k = \overline{ev}_-^k(\gamma_+, \gamma'_+; J_+) : \overline{\mathcal{M}_{J_+}^{\text{ind}=k, \text{cyl}}(\gamma_+, \gamma'_+)/\mathbb{R}} \rightarrow S^{k-1}.$$

from Section 6. Also recall that $\overline{ev}_-^k(v_1, r_{1+}, r_{1-})$ depends on the marker r_{1-} at the negative end. Finally, the asymptotic eigenfunctions f_1, \dots, f_k and the cokernel element Y depend on the parametrization given by (v_0, r_{0+}, r_{0-}) .

Lemma 7.1.6. *For any positive integer k ,*

(1) *if $(\gamma'_+)^{1/k}$ is negative hyperbolic, then $n_{+, \tau_l}^k(\gamma_+, \gamma'_+; \gamma_-)$ is given by*

$$\# \mathcal{M}_{\overline{J}_l}^{\text{ind}=-k, \text{cyl}}(\gamma'_+, \gamma_-) \cdot \# \left(\left(\tilde{ev}_-^k(\gamma_+, \gamma'_+; J_+) \right)^{-1} (\{(0, \dots, 0, \pm 1)\}) \right) \frac{1}{m(\gamma'_+)};$$

(2) *if $(\gamma'_+)^{1/k}$ is positive hyperbolic, then we replace $(0, \dots, 0, \pm 1)$ by $(\pm 1, 0, \dots, 0)$.*

Proof. The proof of Lemma 7.1.6 follows from Proposition 8.7.2 and Section 8.9 A. \square

Now we are in a position to finish the proof of Theorem 7.1.1 in the simple \mathcal{M}^0 case.

Proof of Theorem 7.1.1 in the simple \mathcal{M}^0 case. Let us abbreviate

$$\mathcal{M} = \mathcal{M}_{J_+}^{\text{ind}=k, \text{cyl}}(\gamma_+, \gamma'_+)$$

and denote the restriction of \overline{ev}_-^k to the boundary $\partial(\mathcal{M}/\mathbb{R})$ by $\partial \overline{ev}_-^k$. We assume $(\gamma'_+)^{1/k}$ is negative hyperbolic; the case of $(\gamma'_+)^{1/k}$ positive hyperbolic is similar. By Theorems 6.0.7 and 6.0.8, there exists a generic embedded path $\nu : [0, 1] \rightarrow S^{k-1}$ from $(0, \dots, 0, 1)$ to $(0, \dots, 0, -1)$ such that $\nu \pitchfork \overline{ev}_-^k$ and $\nu \pitchfork \partial \overline{ev}_-^k$. In particular, $\partial \overline{ev}_-^k \cap \nu$ consists of finitely many points.

Let

$$\mathcal{M}_+^{k-1}(\zeta_+, \gamma'_+) = \left\{ w \in \mathcal{M}_{J_+}^{\text{ind}=k-1, \text{cyl}}(\zeta_+, \gamma'_+) \mid \tilde{ev}_-^k(\zeta_+, \gamma'_+; J_+)(w) \in \partial \overline{ev}_-^k \cap \nu \right\}.$$

For $k = 1$ we define

$$K_+^{k-1}(\zeta_+, \gamma_-) = \sum_l \# \mathcal{M}_{J_l}^{\text{ind}=-1, \text{cyl}}(\zeta_+, \gamma_-)$$

and for $k \geq 2$ we define

$$K_+^{k-1}(\zeta_+, \gamma_-) = \sum_l \sum_{\gamma'_+ \in \mathcal{P}_{\alpha_+}^L} \# \mathcal{M}_{J_l}^{\text{ind}=-k, \text{cyl}}(\gamma'_+, \gamma_-) \# (\mathcal{M}_+^{k-1}(\zeta_+, \gamma'_+)/\mathbb{R}) \frac{1}{m(\gamma'_+)}.$$

We then define

$$(7.1.6) \quad K_+(\zeta_+) = \sum_{k=1}^{\infty} \sum_{\gamma_- \in \mathcal{P}_{\alpha_-}^L} K_+^{k-1}(\zeta_+, \gamma_-) \frac{1}{m(\gamma_-)} \cdot \gamma_-.$$

We claim that

$$(7.1.7) \quad (\overline{\partial} \overline{ev}_-^k)^{-1}(\nu) = \left(\coprod_{\zeta_+} \mathcal{M}_+^{k-1}(\zeta_+, \gamma'_+)/\mathbb{R} \times (\mathcal{M}_{J_+}^{\text{ind}=1, \text{cyl}}(\gamma_+, \zeta_+)/\mathbb{R}) \right) / \sim',$$

where \sim' is defined as in Equations (7.1.2) and (7.1.3) with all the subscripts increased by 1. Indeed, “ \supseteq ” follows from the fact that the evaluation map \overline{ev}_-^k only depends on the behavior of holomorphic curves near the negative end; and “ \subseteq ” follows from the fact that generically there is no other boundary component by Theorem 6.0.8. Observe that $\{(0, \dots, 0, \pm 1)\} \cap \partial \overline{ev}_-^k = \emptyset$.

By examining the boundary of the 1-dimensional manifold $(\overline{ev}_-^k)^{-1}(\nu)$,

$$(7.1.8) \quad \begin{aligned} & \# \left((\overline{ev}_-^k)^{-1}(\{(0, \dots, 0, \pm 1)\}) \right) \\ &= \sum_{\zeta_+ \in \mathcal{P}_{\alpha_+}^L} \# (\mathcal{M}_+^{k-1}(\zeta_+, \gamma'_+)/\mathbb{R}) \# (\mathcal{M}_{J_+}^{\text{ind}=1, \text{cyl}}(\gamma_+, \zeta_+)/\mathbb{R}) \frac{1}{m(\zeta_+)}, \end{aligned}$$

where the right-hand side comes from Equation (7.1.7). Then Lemma 7.1.6 and Equation (7.1.8) imply that the first term on the right-hand side of Equation (7.1.4) is equal to $\langle K_+ \partial_+(\gamma_+), \gamma_- \rangle$. This finishes the proof of Theorem 7.1.1, modulo the discussion of orientations from Section 9.2. \square

Non-simple \mathcal{M}^0 case. We explain the necessary modifications when some curve $u \in \mathcal{M}^0$ is multiply-covered.

Using the argument from Step 1 of the proof of Theorem 5.2.1, the curve $u \in \mathcal{M}^0$ is regular and moreover is part of a regular 1-dimensional family. This means that all the curves in the connected component of \mathcal{M}^0 containing u are multiply-covered and regular.

Let $\mathcal{M} = \mathcal{M}_{J_+}^{\text{ind}=k, \text{cyl}}(\gamma_+, \gamma'_+)$ as before and let $\mathcal{M}' \subset \mathcal{M}$ be the subset of curves which are b -fold covers, where $b|k$. We then use the inclusion given by Equation (6.0.9) to argue that for generic J the evaluation map $(ev'_-)^k : \mathcal{M}' \rightarrow \mathbb{R}^k$ does not pass through zero and hence descends to $(\tilde{ev}'_-)^k : \mathcal{M}'/\mathbb{R} \rightarrow S^{k-1}$.

The rest of the argument is the same. Suppose the family \mathcal{M}^0 has boundary which consists of a two-level building $v_0 \cup v_1$, where $\text{ind}(v_0) = -k$, $\text{ind}(v_1) = -k$, v_0 is a k -fold cover, and v_1 is a b -fold cover with $b|k$. As before, the only pairs $v_0 \cup v_1$ that glue satisfy $(\tilde{e}v'_-)^k(v_1) = (0, \dots, 0, \pm 1)$ and we continue the family along some generic $\nu \subset S^{k-1}$ from $(0, \dots, 0, 1)$ to $(0, \dots, 0, -1)$, where the evaluation map we are taking is $\tilde{e}v_-^k : \mathcal{M}/\mathbb{R} \rightarrow S^{k-1}$.

7.2. $HC(\mathcal{D})$ is independent of \mathcal{D} . To show that $HC(\mathcal{D})$ is independent of \mathcal{D} , we study the composition of chain maps induced by a composition of symplectic cobordisms.

For $j = 1, 2$, let (X^j, α^j) be an exact symplectic cobordism as in Section 5.2, with $M_-^1 = M_+^2$ and $\alpha_-^1 = \alpha_+^2$. We define X^{12} to be $X^1 \cup X^2$ identified along $M_-^1 = M_+^2$ and define the 1-form α^{12} on X^{12} by $\alpha^{12}|_{X^1} = \alpha^1$ and $\alpha^{12}|_{X^2} = \alpha^2$.

Let $(\hat{X}^j, \hat{\alpha}^j)$ be the completion of (X^j, α^j) and let \bar{J}^j be an almost complex structure which tames $(\hat{X}^j, \hat{\alpha}^j)$ and restricts to α_\pm^j -tame almost complex structures J_\pm^j at the positive and negative ends. Similarly, let $(\hat{X}^{12}, \hat{\alpha}^{12})$ be the completion of (X^{12}, α^{12}) and let \bar{J}^{12} be a generic almost complex structure on \hat{X}^{12} which tames $(\hat{X}^{12}, \hat{\alpha}^{12})$ and coincides with \bar{J}_+^1 and \bar{J}_-^2 at the positive and negative ends.

Theorem 7.2.1. *Suppose that $(\hat{X}^j, \hat{\alpha}^j, \bar{J}^j)$ and $(\hat{X}^{12}, \hat{\alpha}^{12}, \bar{J}^{12})$ satisfy the assumptions of Theorem 5.2.1. Then $\Phi_{(\hat{X}^{12}, \hat{\alpha}^{12}, \bar{J}^{12})}$ and $\Phi_{(\hat{X}^2, \hat{\alpha}^2, \bar{J}^2)} \circ \Phi_{(\hat{X}^1, \hat{\alpha}^1, \bar{J}^1)}$ induce the same map on homology.*

Proof. The proof of this is essentially the same as the proof of Theorem 7.1.1, as one can construct a 1-parameter family of completed symplectic cobordisms between \hat{X}^{12} and $\hat{X}^1 \cup \hat{X}^2$. \square

Theorem 7.1.1 and Theorem 7.2.1 imply:

Corollary 7.2.2. *$HC(\mathcal{D})$ is independent of the auxiliary data \mathcal{D} .*

8. GLUING

In this section we construct the gluing map that is used in Section 7. In Section 7, we see that a sequence of curves in $\coprod_{0 \leq \tau \leq 1} \mathcal{M}_{\bar{J}^\tau}^{\text{ind}=0, \text{cyl}}(\gamma'_+, \gamma_-)$ can degenerate into a holomorphic building

$$([v_-], [v_+]) \in \left(\mathcal{M}_{\bar{J}^{\tau_0}}^{\text{ind}=-k, \text{cyl}}(\gamma'_+, \gamma_-) \times (\mathcal{M}_{J_+}^{\text{ind}=k, \text{cyl}}(\gamma_+, \gamma'_+)/\mathbb{R}) \right) / \sim$$

for some τ_0 . For convenience we write $\mathcal{M} = \mathcal{M}_{J_+}^{\text{ind}=k, \text{cyl}}(\gamma_+, \gamma'_+)$. For most of this section we are assuming Conditions (C1)–(C4) of the prototypical gluing problem, where $u_0, v_0, v_1, \gamma, \gamma'', \gamma'$ are now called $u_-, v_-, v_+, \gamma_+, \gamma'_+, \gamma_-$. At the end we will treat similar, but slightly different, cases in Section 8.9.

Certainly not every $[v_+] \in \mathcal{M}/\mathbb{R}$ can be glued with $[v_-]$ to give a holomorphic map. In this section we closely follow [HT2] (with the appropriate modifications) and define an obstruction bundle

$$\mathcal{O} \rightarrow [R, \infty) \times \mathcal{M}/\mathbb{R}, \quad R \gg 0,$$

and a section \mathfrak{s} of \mathcal{O} such that $\mathfrak{s}^{-1}(0) \cap (\{T\} \times \mathcal{M}/\mathbb{R})$ are exactly the curves $[v_+]$ that glue with $[v_-]$ for $T \gg 0$. The expressions will be substantially simpler since (α_+, J_+) is L -supersimple. The proofs of the results that carry over with minimal changes will be omitted.

Assume for the moment that all the curves in \mathcal{M} are immersed. In Section 8.8 we will explain how to modify the argument in the general case.

From now on we implicitly choose a smooth section of $\mathcal{M} \rightarrow \mathcal{M}/\mathbb{R}$ and representatives of $([v_-], [v_+])$, and write (v_-, v_+) instead of $([v_-], [v_+])$.

Also, in this section the constant $c > 0$ may change from line to line when we are making estimates.

8.1. Pregluing. Fix a constant $T_0 \gg 0$. Also let $T \gg 2T_0$, which is allowed to vary. Let $v_{+,T}(s, t) := v_+(s - 2T, t)$.

Recall the coordinates (s, t, x, y) on the neighborhood $\mathbb{R} \times \mathbb{R}/\mathbb{Z} \times D_{\delta_0/3}^2$ of $\mathbb{R} \times \gamma'_+$ on which J satisfies (J1) and (J2). For sufficiently large T_0 , $v_-(s, t)$ can be written in terms of these coordinates as $(s, t, \eta_-(s, t))$ on $s \geq T_0$ and $v_{+,T}(s, t)$ can be written as $(s, t, \eta_{+,T}(s, t))$ on $s \leq T$.

Fix constants $0 < h < 1$ and $r \gg h^{-1}$. We take $T_0 > 5r$. Choose a cutoff function $\beta : \mathbb{R} \rightarrow [0, 1]$ such that $\beta(s) = 0$ for $s \leq 0$ and $\beta(s) = 1$ for $s \geq 1$. Let $\beta_{-,T}(s) = \beta(\frac{T-s}{hr})$ and $\beta_{+,T_0}(s) = \beta(\frac{s-T_0}{hr})$.

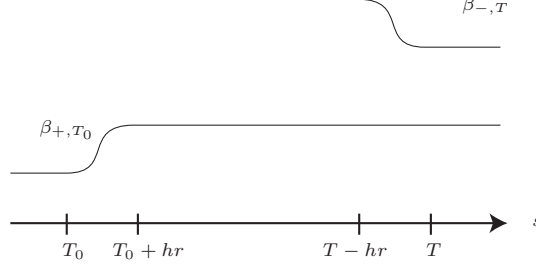


FIGURE 3. Cutoff functions.

We define the *pregluing* of v_+ and v_- by

$$v_*(s, t) = \begin{cases} v_{+,T}(s, t) & \text{for } s \geq T, \\ (s, t, \beta_{+,T_0}(s)\eta_{+,T}(s, t) + \beta_{-,T}(s)\eta_-(s, t)) & \text{for } T_0 \leq s \leq T, \\ v_-(s, t) & \text{for } s \leq T_0. \end{cases}$$

We often suppress the T or T_0 in β_{+,T_0} , $\beta_{-,T}$, and $\eta_{+,T}$.

8.2. Gluing. Let ψ_+ be a section of the normal bundle of $v_{+,T}$ and ψ_- be a section of the normal bundle of v_- . We deform v_* to

$$(8.2.1) \quad v = \exp_{v_*}(\beta_+\psi_+ + \beta_-\psi_-),$$

where \exp_{v_*} is an exponential map which identifies the normal bundle to v_* with a tubular neighborhood of v_* . The exponential maps $\exp_{v_{+,T}}$ and \exp_{v_-} can be

chosen such that

$$\exp_{v_+, T} \psi_+ = (s, t, \eta_+ + \psi_+) \quad \text{and} \quad \exp_{v_-} \psi_- = (s, t, \eta_- + \psi_-)$$

for $s \leq T$ and $s \geq T_0$, respectively. Similarly, we assume that

$$\exp_{v_*}(\beta_+ \psi_+ + \beta_- \psi_-) = (s, t, \eta_* + \beta_+ \psi_+ + \beta_- \psi_-)$$

for $T_0 \leq s \leq T$. Here $\eta_* = \beta_+ \eta_+ + \beta_- \eta_-$.

Let $\tau \in [0, 1]$ be close to τ_0 . The equation $\bar{\partial}_{\bar{J}^\tau} v = 0$ is equivalent to:

$$(8.2.2) \quad \beta_- \left(D_- \psi_- + (\tau - \tau_0) Y' + \frac{\partial \beta_+}{\partial s} \eta_+ + \mathcal{R}_- \right) + \beta_+ \left(D_+ \psi_+ + \frac{\partial \beta_-}{\partial s} \eta_- + \mathcal{R}_+ \right) = 0,$$

where D_- is the linearization of $\bar{\partial}_{\bar{J}^{\tau_0}}$ for the normal bundle of v_- , D_+ is the linearization of $\bar{\partial}_{J_+}$ for the normal bundle of v_+, T , Y' comes from the variation of the almost complex structure from \bar{J}^τ to \bar{J}^{τ_0} , and by abuse of notation we write $\frac{\partial \beta_+}{\partial s} \eta_+$ instead of $\frac{\partial \beta_+}{\partial s} \eta_+ \otimes (ds - idt)$.

We will now describe the terms \mathcal{R}_- and \mathcal{R}_+ . Recall from [HT2, Definition 5.1] that $F(\psi)$ is *type 1 quadratic* if it can be written as

$$F(\psi) = P(\psi) + Q(\psi) \cdot \nabla \psi,$$

where $|P(\psi)| \leq c|\psi|^2$ and $|Q(\psi)| \leq c|\psi|$ for some constant $c > 0$. On $s \leq T_0$,

$$\mathcal{R}_- = F_-(\psi_-, \tau - \tau_0),$$

where $F_-(\psi_-, \tau - \tau_0)$ is type 1 quadratic with respect to $\psi = (\psi_-, \tau - \tau_0)$. On $s \geq T$,

$$\mathcal{R}_+ = F_+(\psi_+),$$

where $F_+(\psi_+)$ is type 1 quadratic. [In local coordinates,

$$\begin{aligned} \frac{\partial(u + \xi)}{\partial s} + J_\tau(u + \xi) \frac{\partial(u + \xi)}{\partial t} &= \left(\frac{\partial u}{\partial s} + J_{\tau_0}(u) \frac{\partial u}{\partial t} \right) + \left(\frac{\partial \xi}{\partial s} + J_{\tau_0}(u) \frac{\partial \xi}{\partial t} \right) \\ &\quad + (J_\tau(u + \xi) - J_{\tau_0}(u)) \frac{\partial u}{\partial t} + (J_\tau(u + \xi) - J_{\tau_0}(u)) \frac{\partial \xi}{\partial t}. \end{aligned}$$

The first term on the right is zero; the second is part of the linearization $D_u \xi$; the third contributes $(\tau - \tau_0) Y'$ and a term $\nabla J_{\tau_0}(\xi) \frac{\partial u}{\partial t}$ towards $D_u \xi$, and the remainder is quadratic and higher in $(\xi, \tau - \tau_0)$; the fourth is bounded by $|(\xi, \tau - \tau_0)| \cdot |\nabla \xi|$.

Claim 8.2.1. *If (α_\pm, J_\pm) is L -supersimple and $T_0 \gg 0$, then $Y' = 0$ on $s \geq T_0$ and we can take \mathcal{R}_\pm to be*

$$(8.2.3) \quad \mathcal{R}_- = \frac{\partial \beta_+}{\partial s} \psi_+, \quad \mathcal{R}_+ = \frac{\partial \beta_-}{\partial s} \psi_-$$

for $s \geq T_0$ and $s \leq T$, respectively.

Proof. Since $\bar{J}^\tau = J_+$ on $s \geq T_0$ and $T_0 \gg 0$, we have $Y' = 0$.

On $T_0 \leq s \leq T$, $\bar{\partial}_{\overline{J}^\tau} v = 0$ can be written as:

$$\begin{aligned} & \beta_- D_- \psi_- + \beta_+ D_+ \psi_+ + \frac{\partial \beta_-}{\partial s} (\eta_- + \psi_-) + \frac{\partial \beta_+}{\partial s} (\eta_+ + \psi_+) \\ &= \beta_- \left(D_- \psi_- + \frac{\partial \beta_+}{\partial s} (\eta_+ + \psi_+) \right) + \beta_+ \left(D_+ \psi_+ + \frac{\partial \beta_-}{\partial s} (\eta_- + \psi_-) \right) = 0, \end{aligned}$$

where $D_- = D_+ = \frac{\partial}{\partial s} - A$. (Note that $D_- \eta_- = 0$ and $D_+ \eta_+ = 0$.) This decomposition is consistent with Equation (8.2.2) and hence we can define \mathcal{R}_\pm by Equation (8.2.3) for $T_0 \leq s \leq T$.

We also define $\mathcal{R}_- = 0$ for $s \geq T$ and $\mathcal{R}_+ = 0$ for $s \leq T_0$. \square

We then rewrite Equation (8.2.2) as

$$\beta_- \Theta_-(\psi_-, \psi_+) + \beta_+ \Theta_+(\psi_-, \psi_+) = 0,$$

by setting

$$(8.2.4) \quad \Theta_-(\psi_-, \psi_+) = D_- \psi_- + (\tau - \tau_0) Y' + \frac{\partial \beta_+}{\partial s} \eta_+ + \mathcal{R}_-,$$

$$(8.2.5) \quad \Theta_+(\psi_-, \psi_+) = D_+ \psi_+ + \frac{\partial \beta_-}{\partial s} \eta_- + \mathcal{R}_+.$$

We want to solve the equations $\Theta_-(\psi_-, \psi_+) = 0$ and $\Theta_+(\psi_-, \psi_+) = 0$, subject to ψ_\pm being in $(\ker(D_\pm))^\perp$ the L^2 -orthogonal complement of $\ker(D_\pm)$. For any sufficiently small ψ_- , one can solve for $\psi_+ = \psi_+(\psi_-) \in (\ker D_+)^\perp$ in $\Theta_+(\psi_-, \psi_+) = 0$, since D_+ is surjective; see Lemma 8.4.1.

Next we solve for ψ_- in $\Theta_-(\psi_-, \psi_+(\psi_-)) = 0$. This is equivalent to solving the following triple of equations

$$(8.2.6) \quad D_- \psi_- + (1 - \Pi) \left((\tau - \tau_0) Y' + \frac{\partial \beta_+}{\partial s} \eta_+ + \mathcal{R}_- \right) = 0,$$

$$(8.2.7) \quad \Pi_Y \left((\tau - \tau_0) Y' + \frac{\partial \beta_+}{\partial s} \eta_+ + \mathcal{R}_- \right) = 0,$$

$$(8.2.8) \quad \Pi' \left(\frac{\partial \beta_+}{\partial s} \eta_+ + \mathcal{R}_- \right) = 0,$$

where Π is the orthogonal projection to $\ker D_-^*$, Π_Y is the orthogonal projection to $\mathbb{R}\langle Y \rangle$, and Π' is the orthogonal projection to $\ker D_-^* \cap Y^\perp$. Here $Y = \Pi Y'$ and Y^\perp is the orthogonal complement of Y . Y is nonzero because $\{\overline{J}^\tau\}_{0 \leq \tau \leq 1}$ is generic.

Given (T, v_+, τ) with $\tau - \tau_0$ small, one can always solve for ψ_+ and ψ_- in Equations (8.2.5) and (8.2.6), subject to $\psi_+ \in (\ker D_+)^\perp$; see Lemma 8.4.2. We then solve for τ in Equation (8.2.7) and set

$$v(T, v_+) := \exp_{v_*}(\beta_+ \psi_+ + \beta_- \psi_-), \quad \tau(T, v_+) := \tau$$

where the right-hand sides of the equations (implicitly) depend on T and v_+ .

8.3. Banach spaces. The function spaces that we use are *Morrey spaces*, following [HT2, Section 5.5].⁷ Let $u : \dot{F} \rightarrow \mathbb{R} \times M$ be an immersed finite energy holomorphic curve and $N \rightarrow \dot{F}$ be a normal bundle. On \dot{F} we choose a Riemannian metric so that the ends are cylindrical and on N we use the metric induced from an \mathbb{R} -invariant Riemannian metric on $\mathbb{R} \times M$.

The *Morrey space* $\mathcal{H}_0(\dot{F}, \Lambda^{0,1} T^* \dot{F} \otimes N)$ is the Banach space which is the completion of the compactly supported sections of $\Lambda^{0,1} T^* \dot{F} \otimes N$ with respect to the norm

$$\|\xi\| = \left(\int_{\dot{F}} |\xi|^2 \right)^{1/2} + \left(\sup_{x \in \dot{F}} \sup_{\rho \in (0,1]} \rho^{-1/2} \int_{B_\rho(x)} |\xi|^2 \right)^{1/2},$$

where $B_\rho(x) \subset \dot{F}$ is the ball of radius ρ about x . Similarly, $\mathcal{H}_1(\dot{F}, N)$ is the completion of the compactly supported sections of N with respect to

$$\|\xi\|_* = \|\nabla \xi\| + \|\xi\|.$$

The analog of the usual Sobolev embedding theorem is the following:

Lemma 8.3.1. *There is a bounded linear map*

$$\mathcal{H}_1(\dot{F}, N) \rightarrow C^{0,1/4}(\dot{F}, N), \quad \xi \mapsto \xi,$$

where $C^{0,1/4}$ denotes the space of Hölder continuous functions with exponent $\frac{1}{4}$.

8.4. Some estimates. Let $\lambda = \min\{\lambda_1, |\lambda_{-1}|\}$. Let \mathcal{H}_+ be the L^2 -orthogonal complement of $\ker D_+$ in $\mathcal{H}_1(\dot{F}, N)$ corresponding to v_+ , \mathcal{H}_- be $\mathcal{H}_1(\dot{F}, N)$ corresponding to v_- , and \mathcal{B}_\pm be the closed ball of radius ε in \mathcal{H}_\pm centered at 0.

The following lemma closely follows [HT2, Proposition 5.6], but the estimates are slightly different.

Lemma 8.4.1. *There exist $r \gg 0$ and $\varepsilon > 0$ such that for $T \gg 0$ the following holds:*

- (1) *There is a map $P : \mathcal{B}_- \rightarrow \mathcal{H}_+$ such that $\Theta_+(\psi_-, P(\psi_-)) = 0$.*
- (2) *$\|P(\psi_-)\|_* \leq cr^{-1}(e^{-\lambda T} + \|\psi_-\|_*)$.*

Proof. (1) We are trying to solve for

$$D_+ \psi_+ + \frac{\partial \beta_-}{\partial s} (\eta_- + \psi_-) + \mathcal{R}'_+(\psi_+) = 0,$$

where $\mathcal{R}'_+(\psi_+)$ is type 1 quadratic. Writing

$$(8.4.1) \quad \mathcal{I}(\psi_+) = -D_+^{-1} \left(\frac{\partial \beta_-}{\partial s} (\eta_- + \psi_-) + \mathcal{R}'_+(\psi_+) \right),$$

where D_+^{-1} is the bounded inverse of $D_+|_{\mathcal{H}_+}$, we define $P(\psi_-)$ as the unique fixed point $\mathcal{I}(\psi_+) = \psi_+$.

⁷This is rather nonstandard and we chose to adopt it to avoid redoing the work in [HT2] for $W^{k,p}$ -spaces.

The unique fixed point is guaranteed by the contraction mapping theorem, which relies on two estimates. Our first estimate is

$$(8.4.2) \quad \|\mathcal{I}(\psi_+)\|_* \leq cr^{-1}(e^{-\lambda T} + \|\psi_-\|_*) + c\|\psi_+\|_*^2.$$

Indeed, since $|\frac{\partial \beta_-}{\partial s}| < cr^{-1}$ for some $c > 0$,

$$\left\| \frac{\partial \beta_-}{\partial s}(\eta_- + \psi_-) \right\| \leq cr^{-1}(e^{-\lambda T} + \|\psi_-\|_*),$$

and, since \mathcal{R}'_+ is type 1 quadratic,

$$\|\mathcal{R}'_+(\psi_+)\| \leq c\|\psi_+\|_*^2.$$

The estimate follows from the boundedness of D_+^{-1} ; observe that the domain of D_+^{-1} uses the norm $\|\cdot\|$ and the range of D_+^{-1} uses $\|\cdot\|_*$. If $r, T \gg 0$ and $\varepsilon > 0$ is small, then the right-hand side of Equation 8.4.2 is $< \varepsilon$ whenever $\|\psi_-\|_*, \|\psi_+\|_* < \varepsilon$. Hence \mathcal{I} maps a radius ε ball into itself.

Our second estimate is

$$(8.4.3) \quad \begin{aligned} \|\mathcal{I}(\psi_+) - \mathcal{I}(\psi'_+)\|_* &= \|D_+^{-1}(\mathcal{R}'_+(\psi_+) - \mathcal{R}'_+(\psi'_+))\|_* \\ &\leq c(\|\psi_+\|_* + \|\psi'_+\|_*) \cdot \|\psi_+ - \psi'_+\|_*. \end{aligned}$$

This follows from observing that D_+^{-1} is bounded and \mathcal{R}' is type 1 quadratic. When $\|\psi_+\|_*, \|\psi'_+\|_*$ are sufficiently small, \mathcal{I} gives a contraction mapping. This proves (1).

(2) $\psi_+ = P(\psi_-)$ satisfies $\mathcal{I}(\psi_+) = \psi_+$. Hence Equation (8.4.2) gives:

$$\|\psi_+\|_* \leq cr^{-1}(e^{-\lambda T} + \|\psi_-\|_*) + c\|\psi_+\|_*^2.$$

This implies (2) since $c\|\psi_+\|_*^2 \ll \|\psi_+\|_*$ for $\varepsilon > 0$ small. \square

The following lemma closely follows [HT2, Proposition 5.7].

Lemma 8.4.2. *There exist $r \gg 0$, $\varepsilon > 0$, and $\varepsilon_0 > 0$ such that for $T \gg 0$ the following holds:*

- (1) *There is a map $P' : \mathcal{B}_+ \times [-\varepsilon_0, \varepsilon_0] \rightarrow \mathcal{H}_-$ such that Equation (8.2.6) holds with $\psi_- = P'(\psi_+, \tau - \tau_0)$.*
- (2) $\|P'(\psi_+, \tau - \tau_0)\|_* \leq cr^{-1}(e^{-\lambda T} + \|\psi_+\|_*) + c|\tau - \tau_0|$.

Here $\tau - \tau_0$ is the coordinate for $[-\varepsilon_0, \varepsilon_0]$.

Proof. (1) We are trying to solve for

$$D_-\psi_- + (1 - \Pi) \left((\tau - \tau_0)Y' + \frac{\partial \beta_+}{\partial s}(\eta_+ + \psi_+) + \mathcal{R}'_-(\psi_-, \tau - \tau_0) \right) = 0,$$

where \mathcal{R}'_- is type 1 quadratic. Let us write

$$\mathcal{I}_{\tau - \tau_0}(\psi_-) = -D_-^{-1}(1 - \Pi) \left((\tau - \tau_0)Y' + \frac{\partial \beta_+}{\partial s}(\eta_+ + \psi_+) + \mathcal{R}'_-(\psi_-, \tau - \tau_0) \right).$$

We first estimate

(8.4.4)

$$\|\mathcal{I}_{\tau-\tau_0}(\psi_-)\|_* \leq c|\tau - \tau_0| + cr^{-1}(e^{-\lambda T} + \|\psi_+\|_*) + c(\|\psi_-\|_*^2 + |\tau - \tau_0|^2).$$

Provided $r, T \gg 0$, $\varepsilon > 0$ is small, and $|\tau - \tau_0|$ is small, the right-hand side is $< \varepsilon$ whenever $\|\psi_-\|_*, \|\psi_+\|_* < \varepsilon$. Hence $\mathcal{I}_{\tau-\tau_0}$ maps a radius ε ball into itself.

Next we estimate

$$(8.4.5) \quad \|\mathcal{I}_{\tau-\tau_0}(\psi_-) - \mathcal{I}_{\tau-\tau_0}(\psi'_-)\|_* = c\|\mathcal{R}'_-(\psi_-, \tau - \tau_0) - \mathcal{R}'_-(\psi'_-, \tau - \tau_0)\| \\ \leq C\|\psi_- - \psi'_-\|_*, \quad 0 < C < 1,$$

provided $|\tau - \tau_0|, \|\psi_-\|_*$, and $\|\psi'_-\|_*$ are small. The above estimates provide a contraction mapping.

(2) Follows from Equation (8.4.4) and $\mathcal{I}_{\tau-\tau_0}(\psi_-) = \psi_-$. \square

We also have:

Lemma 8.4.3. *There exist $r \gg 0$ and $\varepsilon > 0$ such that for $T \gg 0$ the following holds:*

- (1) *There is a map $P'' : \mathcal{B}_- \times \mathcal{B}_+ \rightarrow [-\varepsilon_0, \varepsilon_0]$ such that Equation (8.2.7) holds with $\tau - \tau_0 = P''(\psi_-, \psi_+)$.*
- (2) $|P''(\psi_-, \psi_+)| \leq cr^{-1}(e^{-\lambda T} + \|\psi_+\|_*) + c\|\psi_-\|_*$.

Proof. This is proved using the contraction mapping theorem as in the proofs of Lemmas 8.4.1 and 8.4.2 and is left to the reader. \square

Putting Lemmas 8.4.1, 8.4.2, and 8.4.3 together we obtain:

Lemma 8.4.4. *There exist $r \gg 0$, $\varepsilon > 0$, and $\varepsilon_0 > 0$ such that for $T \gg 0$ there is a unique solution $(\psi_-, \psi_+, \tau - \tau_0) \in \mathcal{B}_- \times \mathcal{B}_+ \times [-\varepsilon_0, \varepsilon_0]$ to the equation $\Theta_+(\psi_-, \psi_+) = 0$ and Equations (8.2.6) and (8.2.7). Moreover,*

$$\|\psi_-\|_*, \|\psi_+\|_*, |\tau - \tau_0| \leq cr^{-1}e^{-\lambda T}.$$

8.5. Obstruction bundle and obstruction section. Let \mathcal{O} be the obstruction bundle

$$\mathcal{O} \rightarrow [R, \infty) \times \mathcal{M}/\mathbb{R}, \quad R \gg 0,$$

whose fiber over (T, v_+) is

$$\mathcal{O}_{T, v_+} = \text{Hom}(\ker D_-^*/\mathbb{R}\langle Y \rangle, \mathbb{R}).$$

We define the section \mathfrak{s} of \mathcal{O} as follows:

$$(8.5.1) \quad \mathfrak{s}(T, v_+)(\sigma) = \left\langle \sigma, \frac{\partial \beta_+}{\partial s} \eta_+ + \mathcal{R}_- \right\rangle,$$

where $\sigma \in \ker D_-^*/\mathbb{R}\langle Y \rangle$ and $\langle \cdot, \cdot \rangle$ denotes the L^2 -inner product. More precisely, given (T, v_+) , we solve for ψ_+, ψ_-, τ in $\Theta_+(\psi_-, \psi_+) = 0$ and in Equations (8.2.6) and (8.2.7) and evaluate the right-hand side of Equation (8.5.1) using these ψ_+, ψ_- , and τ .

Theorem 8.5.1 ([HT2, Lemma 6.3]). *\mathfrak{s} is smooth for $R \gg 0$.*

As in Lemma 1.0.5,

$$\dim \ker(D_-) = 0, \quad \dim \ker(D_-^*) = \dim \operatorname{coker}(D_-) = k.$$

Therefore, we expect $\mathfrak{s}^{-1}(0)$ to intersect $\{T\} \times \mathcal{M}/\mathbb{R}$ at finitely many points, for generic $T \geq 10r$.

To any $(T, v_+) \in [R, \infty) \times \mathcal{M}/\mathbb{R}$ we can associate $v(T, v_+)$ and $\tau(T, v_+)$, as described at the end of Section 8.2. When $(T, v_+) \in \mathfrak{s}^{-1}(0)$, we can view $(v(T, v_+), \tau(T, v_+))$ as an element of $\coprod_{0 \leq \tau \leq 1} \mathcal{M}_{\overline{J}^\tau}^{\operatorname{ind}=0, \operatorname{cyl}}(\gamma'_+, \gamma_-)$. Hence we have the gluing map

$$G : \mathfrak{s}^{-1}(0) \rightarrow \coprod_{0 \leq \tau \leq 1} \mathcal{M}_{\overline{J}^\tau}^{\operatorname{ind}=0, \operatorname{cyl}}(\gamma'_+, \gamma_-),$$

$$(T, v_+) \mapsto (v(T, v_+), \tau(T, v_+)).$$

8.6. Bijectivity of the gluing map. Let K be a subset of \mathcal{M}/\mathbb{R} . Given $\delta > 0$, let $\tilde{\mathcal{G}}_\delta(v_-, K)$ be the set of (not necessarily holomorphic) maps that are δ -close to some holomorphic building $v_- \cup v_+$ in the sense of [HT2, Definition 7.1], where $v_+ \in K$. Let $\mathcal{G}_\delta(v_-, K)$ be the subset of

$$\coprod_{0 \leq \tau \leq 1} \mathcal{M}_{\overline{J}^\tau}^{\operatorname{ind}=0, \operatorname{cyl}}(\gamma_+, \gamma_-)$$

consisting of pairs (v, τ) , where v is a \overline{J}^τ -holomorphic curve in $\tilde{\mathcal{G}}_\delta(v_-, K)$. Also let $\mathcal{U}_{\delta, K} \subset [R, \infty) \times K$ be the set of (T, v_+) such that $v(T, v_+) \in \tilde{\mathcal{G}}_\delta(v_-, K)$.

Theorem 8.6.1 ([HT2, Theorem 7.3]). *Suppose $K \subset \mathcal{M}/\mathbb{R}$ is compact.*

- (1) *For $R' > 0$ sufficiently large with respect to δ , $[R', \infty) \times K \subset \mathcal{U}_{\delta, K}$.*
- (2) *For $\delta > 0$ sufficiently small, there exists a compact set $K \subset K' \subset \mathcal{M}/\mathbb{R}$ such that the restriction*

$$G|_{\mathcal{U}_{\delta, K'}} : \mathfrak{s}^{-1}(0) \cap \mathcal{U}_{\delta, K'} \rightarrow \mathcal{G}_\delta(v_-, \mathcal{M}/\mathbb{R})$$

of G satisfies $\operatorname{Im}(G|_{\mathcal{U}_{\delta, K'}}) \supset \mathcal{G}_\delta(v_-, K)$. Moreover, $G|_{\mathcal{U}_{\delta, K'}}$ is a homeomorphism onto its image.

8.7. The linearized section \mathfrak{s}_0 . Recall the evaluation map $\tilde{e}v_-^k : \mathcal{M}/\mathbb{R} \rightarrow S^{k-1}$ and also that $\dim \mathcal{M}/\mathbb{R} = k - 1$.

We define a homotopy of sections \mathfrak{s}_ζ , $\zeta \in [0, 1]$, as follows: For $\zeta \in [\frac{1}{2}, 1]$,

$$(8.7.1) \quad \mathfrak{s}_\zeta(T, v_+)(\sigma) = \left\langle \sigma, \frac{\partial \beta_+}{\partial s} \eta_+ + \beta(2\zeta - 1) \cdot \mathcal{R}_- \right\rangle,$$

where β is the cutoff function from Section 8.1. More precisely, given (T, v_+) , we solve for ψ_+ , ψ_- , τ in $\Theta_+(\psi_-, \psi_+) = 0$ and in Equations (8.2.6) and (8.2.7) with \mathcal{R}_- replaced by $\beta(2\zeta - 1)\mathcal{R}_-$, and evaluate the right-hand side of Equation (8.7.1) using these ψ_+ , ψ_- , and τ . Observe that the estimates from Section 8.4 carry over with \mathcal{R}_- replaced by $\beta(2\zeta - 1)\mathcal{R}_-$, allowing us to define \mathfrak{s}_ζ . For $\zeta \in [0, \frac{1}{2}]$,

$$(8.7.2) \quad \mathfrak{s}_\zeta(T, v_+)(\sigma) = \left\langle \sigma_\zeta, \frac{\partial \beta_+}{\partial s} \eta_+ \right\rangle,$$

where σ_ζ is the linear interpolation between σ and $\tilde{\sigma}$, and $\tilde{\sigma}$ is σ with the $f_{k+1}(t), f_{k+2}(t), \dots$ terms truncated. Note that we are only concerned with the positive end of σ .

Here $\mathfrak{s}_1 = \mathfrak{s}$ and $\mathfrak{s}_0(T, v_+)(\sigma) = \left\langle \tilde{\sigma}, \frac{\partial \beta_+}{\partial s} \eta_+ \right\rangle$.

Lemma 8.7.1. *With respect to the basis $\{\sigma_1, \dots, \sigma_k\}$ from Lemma 1.0.5,*

$$(8.7.3) \quad \mathfrak{s}_0(T, v_+)(\sigma_i) = e^{-2\lambda_i T} c_i$$

where $\tilde{e}v_-^k(v_+) = (c_1, \dots, c_k)$ and $i = 1, \dots, k-1$, and

$$(8.7.4) \quad \mathfrak{s}_0^{-1}(0) = [R, \infty) \times (\tilde{e}v_-^k)^{-1}(\{(0, \dots, 0, \pm 1)\}).$$

Proof. We compute

$$\begin{aligned} \mathfrak{s}_0(T, v_+)(\sigma_i) &= \left\langle \tilde{\sigma}_i, \frac{\partial \beta_+}{\partial s} \eta_+ \right\rangle \\ &= \left\langle e^{-\lambda_i s} f_i(t), \frac{\partial \beta_+}{\partial s} \left(\sum_j c_j e^{\lambda_j(s-2T)} f_j(t) \right) \right\rangle \\ &= e^{-2\lambda_i T} c_i. \end{aligned}$$

Recall that $\{f_j\}_{j \in \mathbb{Z} - \{0\}}$ is an L^2 -orthonormal basis and that $\int_{-\infty}^{\infty} \frac{\partial \beta_+}{\partial s} ds = 1$. The calculation of $\mathfrak{s}_0^{-1}(0)$ immediately follows from Equation (8.7.3). \square

The following proposition allows us to substitute \mathfrak{s} for the linearized section \mathfrak{s}_0 :

Proposition 8.7.2. *Let $K \subset \mathcal{M}/\mathbb{R}$ be a compact $(k-1)$ -dimensional submanifold with boundary such that $\tilde{e}v_-^k(\partial K) \cap \{(0, \dots, 0, \pm 1)\} = \emptyset$. Then for $R \gg 0$, there are no zeros of \mathfrak{s}_ζ , $\zeta \in [0, 1]$, on $[R, \infty) \times \partial K$.*

Proof. For our purposes we take a smooth section of $\mathcal{M} \rightarrow \mathcal{M}/\mathbb{R}$ over K and a constant $C > 0$ so that, for each v_+ representing $[v_+] \in K$,

(K1) $v_+|_{s \leq 0}$ has image in $\mathbb{R} \times (\mathbb{R}/\mathbb{Z}) \times D_{\delta_0/3}^2$ and is graphical over a subset of $\mathbb{R} \times \gamma'_+$;

(K2) if we write $v_+ = \left(s, t, \sum_{i>0} c_i e^{\lambda_i s} f_i(t) \right)$ on $s \leq 0$, then $\sum_{i=1}^k c_i^2 = C^2 \delta_0^2$;

and
(K3) $\sum_{i=1}^k c_i^2 \gg \sum_{i>k} c_i^2$.

We then take $0 < \delta \ll C \delta_0$.

Let $R > 0$ be sufficiently large with respect to δ . Arguing by contradiction, suppose $(T, v_+) \in \mathfrak{s}^{-1}(0) \cap ([R, \infty) \times \partial K)$. Given $(T, v_+) \in [R, \infty) \times \partial K$, consider $(v(T, v_+), \tau(T, v_+))$. Observe that $v(T, v_+) = \exp_{v_*}(\psi_+)$ for $s \geq 2T$; $v(T, v_+) = \exp_{v_*}(\psi_-)$ for $s \leq T_0$; and $v(T, v_+)$ can be written as

$$(s, t, \beta_+(\eta_+ + \psi_+) + \beta_-(\eta_- + \psi_-))$$

on $T_0 \leq s \leq 2T$ by our choice of section. Recall that $v_{+,T}$ is v_+ translated up by $2T$.

By Lemmas 8.4.4 and 8.3.1,

$$(8.7.5) \quad |\psi_-|_{C^0}, |\psi_+|_{C^0}, |\tau - \tau_0| < cr^{-1}e^{-\lambda T},$$

for some constant $c > 0$. In view of the normalization of v_+ along $s = 0$ given by (K2) and (K3), we have

$$(8.7.6) \quad |\psi_+|_{C^0} \ll |\eta_+|_{C^0}$$

along $s = 2T$, provided $R \gg 0$.

We now make some explicit calculations in local coordinates for ψ_+ on the $s \leq 2T$ region. By Claim 8.2.1, ψ_+ satisfies the equation

$$(8.7.7) \quad D_+\psi_+ + \frac{\partial\beta_-}{\partial s}(\eta_- + \psi_-) = 0.$$

For the term $\eta_- + \psi_-$ in Equation (8.7.7) we have:

Claim 8.7.3. *If $(T, v_+) \in \mathfrak{s}^{-1}(0)$, then*

$$\eta_- + \psi_- = \sum_{i < 0} d_i e^{\lambda_i s} f_i(t)$$

for $s \geq T_0 + hr$.

Proof of Claim 8.7.3. First note that η_- is holomorphic. On $s \geq T_0$, ψ_- satisfies the equation

$$D_-\psi_- + \frac{\partial\beta_+}{\partial s}(\eta_+ + \psi_+) = 0$$

by Claim 8.2.1. Restricted to $s \geq T_0 + hr$, this becomes $D_-\psi_- = 0$. Since both η_- and ψ_- have exponential decay as $s \rightarrow \infty$, $\eta_- + \psi_-$ can be written as $\sum_{i < 0} d_i e^{\lambda_i s} f_i(t)$ for $s \geq T_0 + hr$. \square

Next let us write

$$\psi_+(s, t) = \sum_{i \in \mathbb{Z} - \{0\}} b_i(s) e^{\lambda_i s} f_i(t).$$

Then we have the following:

Claim 8.7.4. *If $(T, v_+) \in \mathfrak{s}^{-1}(0)$, then on $s < 2T$:*

- (1) *for $i > 0$, $b_i(s)$ is constant; and*
- (2) *for $i < 0$, $b_i(s) = -d_i \int_{-\infty}^s \frac{\partial\beta_-}{\partial s} ds$.*

Proof of Claim 8.7.4. Since ψ_+ has exponential decay as $s \rightarrow -\infty$ and $\frac{\partial\beta_-}{\partial s} = 0$ on $s < T - hr$, $b_i(s)$ must be constant for all i and equal to zero for $i < 0$ on $s < T - hr$ by Equation (8.7.7). By a similar argument which uses Claim 8.7.3, $b_i(s)$ is constant on $T_0 + hr < s < 2T$ for $i > 0$. Therefore, $b_i(s)$ is constant on $s < 2T$ for $i > 0$, which gives (1). Solving for the Fourier coefficients for $i < 0$ gives (2). \square

Claim 8.7.5. *If $R \gg 0$, then $\tilde{e}v_-^k(\eta_+ + \psi_+)$ is close to $\tilde{e}v_-^k(\eta_+)$ for all $[T, v_+) \in [R, \infty) \times \partial K$. In particular, $\tilde{e}v_-^k(\eta_+ + \psi_+) \neq (0, \dots, 0, \pm 1)$.*

We will be using the identification $\mathbb{R}^k/\mathbb{R}^+ \simeq S_{C\delta_0}^{k-1}$; see Remark 6.0.3. This is to take advantage of (K2).

Proof of Claim 8.7.5. If we write $\eta_+(s, t) = \sum_{i>0} c_i e^{\lambda_i(s-2T)} f_i(t)$, then the order k Fourier polynomial for the negative end of $\eta_+ + \psi_+$ is

$$P_k(\eta_+ + \psi_+) = \sum_{i=1}^k (c_i e^{-2\lambda_i T} + b_i) e^{\lambda_i s} f_i(t).$$

By (K2), $\tilde{e}v_-^k(\eta_+) = (c_1, \dots, c_k)$. By using Equation (8.7.6) along $s = 2T$, we see that $\sum_{i=1}^k c_i^2 \gg \sum_{i=1}^k (b_i e^{2\lambda_i T})^2$. The claim then follows. \square

Claim 8.7.6. *For any $\varepsilon > 0$, there exists $R \gg 0$ such that*

$$|(\eta_+ + \psi_+) - P_k(\eta_+ + \psi_+)| < \varepsilon |P_k(\eta_+ + \psi_+)|$$

pointwise on $s \leq 2T$ for all $[T, v_+) \in [R, \infty) \times \partial K$.

Proof of Claim 8.7.6. By (K3),

$$|(\eta_+ + \psi_+) - P_k(\eta_+ + \psi_+)| < \varepsilon |P_k(\eta_+ + \psi_+)|$$

holds along $s = 2T$. Since the error $(\eta_+ + \psi_+) - P_k(\eta_+ + \psi_+)$ has faster exponential decay than $P_k(\eta_+ + \psi_+)$ as s decreases, the claim follows. \square

Let us now consider $\Theta_-(\psi_-, \psi_+) = 0$, which can be written as

$$(8.7.8) \quad \bar{\partial}_{\bar{J}^r} \exp_{v_-}(\psi_-) = -\frac{\partial \beta_+}{\partial s}(\eta_+ + \psi_+).$$

Consider the finite-dimensional vector space $W = \mathbb{R}\langle \frac{\partial \beta_+}{\partial s} e^{\lambda_i s} f_i(t) \rangle_{i=1}^k$. We write $Z_k = \mathbb{R}\langle \frac{\partial \beta_+}{\partial s} e^{\lambda_k s} f_k(t) \rangle$.

Lemma 8.7.7. *If $B \subset W$ is a sufficiently small ball centered at 0, then for all $g \in B - Z_k$ there is no solution to the equation*

$$(8.7.9) \quad \bar{\partial}_{\bar{J}^r} \exp_{v_-}(\psi_-) = g.$$

The same is also true for all g of the form $g' + g''$, where $g' \in B - Z_k$, $g'' \in \mathcal{H}_0(\dot{F}, \Lambda^{0,1} T^ F \otimes N)$, $\|g'\| < \varepsilon \|g''\|$, and $\varepsilon > 0$ is a fixed sufficiently small number.*

Proof of Lemma 8.7.7. First we claim that if $g \in B \cap Z_k$, then there exists a pair (ψ_+, τ) which solves Equation (8.7.9): Let us write $v_- = u_- \circ \pi$. Then $g \in B \cap Z_k$ can be written as $g = \pi^* g_0$ for some g_0 . There is a solution $(\psi_{-,0}, \tau)$ of

$$\bar{\partial}_{\bar{J}^r} \exp_{u_-}(\psi_{-,0}) = g_0,$$

since $\ker(D_{u_-}^*)$ is 1-dimensional and is generated by Y_0 which comes from the variation of \bar{J}^r . If we write $\psi_- = \pi^* \psi_{-,0}$, then the pair (ψ_-, τ) solves Equation (8.7.9).

The claim, interpreted suitably, implies the lemma: Consider the map

$$\begin{aligned}\bar{\partial}^{\mathcal{P}} : \mathcal{H}_1(\dot{F}, N) \oplus \mathbb{R} &\rightarrow \mathcal{H}_0(\dot{F}, \Lambda^{0,1}T^*\dot{F} \otimes N), \\ (\psi_-, \tau - \tau_0) &\mapsto \mathcal{P}(\bar{\partial}_{\mathcal{T}^\tau} \exp_{v_-}(\psi_-)),\end{aligned}$$

where \mathcal{P} refers to a suitable parallel transport. For a sufficiently small ball \mathcal{B} of $\mathcal{H}_1(\dot{F}, N) \oplus \mathbb{R}$ centered at $(0, 0)$, $\bar{\partial}^{\mathcal{P}}(\mathcal{B})$ is a submanifold of codimension $k - 1$ in $\mathcal{H}_0(\dot{F}, \Lambda^{0,1}T^*\dot{F} \otimes N)$: this is because the linearization $L_{(\psi_-, \tau - \tau_0)}$ of $\bar{\partial}^{\mathcal{P}}$ at any point $(\psi_-, \tau - \tau_0) \in \mathcal{B}$ has Fredholm index $-k + 1$, $L_{(0,0)}$ has maximal rank, i.e., codimension $k - 1$, and hence $L_{(\psi_-, \tau - \tau_0)}$ also has codimension $k - 1$. If $(\psi_-, \tau - \tau_0) \in \mathcal{B}$, then $L_{(\psi_-, \tau - \tau_0)}$ has image which is close to that $L_{(0,0)}$. Since

$$L_{(0,0)} = T_{(0,0)}\bar{\partial}^{\mathcal{P}}(\mathcal{B}) = \text{Im}(D_-) + \mathbb{R}\langle \sigma_k \rangle,$$

the tangent space $T_{(\psi_-, \tau - \tau_0)}\bar{\partial}^{\mathcal{P}}(\mathcal{B})$ is close to $\text{Im}(D_-) + \mathbb{R}\langle \sigma_k \rangle$ when $(\psi_-, \tau - \tau_0)$ is small. If we write $g = g^b + g^\sharp \in B - Z_k$ with $g^b \in Z_k$ and $0 \neq g^\sharp \in \mathbb{R}\langle \frac{\partial \beta_+}{\partial s} e^{\lambda_i s} f_i(t) \rangle_{i=1}^{k-1}$ and solve for $(\psi_-, \tau - \tau_0)$ in $\bar{\partial}_{\mathcal{T}^\tau} \exp_{v_-}(\psi_-) = g^b$, then the absolute value of the angle between g^\sharp and $T_{(\psi_-, \tau - \tau_0)}\bar{\partial}^{\mathcal{P}}(\mathcal{B})$ is bounded below by a positive constant. Hence Equation (8.7.9) does not admit a solution for $g \in B - Z_k$. \square

By Claims 8.7.5 and 8.7.6, the right-hand side of Equation (8.7.8) is an element of W , plus a small error. Hence Equation (8.7.8) has no solution by Lemma 8.7.7. Unwinding the definition of \mathfrak{s} , $\mathfrak{s}(T, v_+) \neq 0$ and we have a contradiction.

The argument for \mathfrak{s}_ζ , $\zeta \in [\frac{1}{2}, 1]$, is done in exactly the same way. The estimates from Lemmas 8.4.1(2) and 8.4.2(2) carry over and the nonexistence of the solution to Equation (8.7.8) also holds for $\Theta_-(\psi_-, \psi_+) = 0$ with \mathcal{R}_- replaced by $\beta(2\zeta - 1)\mathcal{R}_-$.

Finally, \mathfrak{s}_ζ , $\zeta \in [0, \frac{1}{2}]$, is close to \mathfrak{s}_0 for $R \gg 0$. Hence there are no zeros of \mathfrak{s}_ζ , $\zeta \in [0, \frac{1}{2}]$, on $[R, \infty) \times \partial K$. This completes the proof of Proposition 8.7.2. \square

8.8. The general prototypical gluing. So far we have been assuming that all the curves in \mathcal{M} are immersed. In general we have the following proposition, which allows us to reduce to the immersed case.

Proposition 8.8.1. *For $\delta > 0$ sufficiently small, there are no holomorphic curves in $\coprod_{0 \leq \tau \leq 1} \mathcal{M}_{\mathcal{T}^\tau}^{\text{ind}=0, \text{cyl}}(\gamma_+, \gamma_-)$ that are:*

- (1) δ -close to (v_-, v_1, \dots, v_l) , where $v_1 \cup \dots \cup v_l$ is an l -level building of $\partial(\mathcal{M}/\mathbb{R})$ and $l > 1$ or
- (2) δ -close to (v_-, v_+) , where $v_+ \in \mathcal{M}^{\text{sing}}$.

Remark 8.8.2. The proof of Proposition 8.8.1(1) does not require any v_i to be regular.

Proof. (1) Suppose there is a curve u satisfying (1). We will apply the proof of Proposition 8.7.2 to obtain a contradiction. Let v_+ be a pregluing of $v_1 \cup \dots \cup v_l$ with gluing parameters T_1, \dots, T_{l-1} and let v_* be a pregluing of $v_- \cup v_+$ with

gluing parameter T . We focus on the neck portion $T_0 \leq s \leq 2T$ between v_- and $v_{+,T}$, which we take to be preglued in the same way as in Sections 8.1 and 8.2.

Let ψ_- be a section of the normal bundle of v_- , defined on $s \leq T_0$, so that $u = \exp_{v_-}(\psi_-)$. Similarly, let ψ_+ be a section of the normal bundle of $v_{+,T}$, defined on $s \geq 2T$, so that $u = \exp_{v_{+,T}}(\psi_+)$. (Assume without loss of generality that v_+ is immersed, since we only care about the negative end of v_+ .) If $\delta > 0$ is sufficiently small, then the sections ψ_- and ψ_+ exist and u is graphical over $[T_0, 2T] \times \mathbb{R}/\mathbb{Z}$ on the neighborhood $[T_0, 2T] \times \mathbb{R}/\mathbb{Z} \times D_{\delta_0/3}^2$ with coordinates (s, t, x, y) . Also let η_+, η_- be as defined before.

Claim 8.8.3. *There exist extensions of ψ_+ to $s \leq 2T$ and ψ_- to $s \geq T_0$, viewed as functions to $D_{\delta_0/3}^2$, such that*

$$\begin{aligned}\Theta_-(\psi_-, \psi_+) &= D_-(\eta_- + \psi_-) + \frac{\partial \beta_+}{\partial s}(\eta_+ + \psi_+) = 0, \\ \Theta_+(\psi_-, \psi_+) &= D_+(\eta_+ + \psi_+) + \frac{\partial \beta_-}{\partial s}(\eta_- + \psi_-) = 0,\end{aligned}$$

on $s \geq T_0$ and $s \leq 2T$, respectively.

Proof. This is proved in Lemmas 7.6 and 7.7 of [HT2].

More directly, if

$$u(s, t) = \sum_{i \in \mathbb{Z} - \{0\}} c_i e^{\lambda_i s} f_i(t)$$

on $T_0 + hr \leq s \leq T - hr$, then we set

$$\eta_+ + \psi_+ = \sum_{i > 0} c_i e^{\lambda_i s} f_i(t), \quad s \leq T - hr,$$

$$\eta_- + \psi_- = \sum_{i < 0} c_i e^{\lambda_i s} f_i(t), \quad s \geq T_0 + hr.$$

We solve $\Theta_+(\psi_-, \psi_+) = 0$ on $T - hr \leq s \leq 2T$ by decomposing into eigenmodes: Writing

$$\eta_+ + \psi_+ = \sum_{i \in \mathbb{Z} - \{0\}} c_i(s) e^{\lambda_i s} f_i(t),$$

$c_i(s) = c_i$ for $i > 0$. On the other hand, for $i < 0$, we have $c'_i(s) + \frac{\partial \beta_-}{\partial s} c_i = 0$ and the solution $c_i(s) = -c_i \int_{-\infty}^s \frac{\partial \beta_-}{\partial s} ds$ has the right boundary conditions $c_i(T - hr) = 0$ and $c_i(2T) = c_i$. The equation $\Theta_-(\psi_-, \psi_+)$ is solved similarly. \square

Returning to the proof of Proposition 8.8.1, now that we have sections ψ_- and ψ_+ irrespective of the regularity of v_i , we reconsider Equation (8.7.8), which is satisfied by the pair (ψ_-, ψ_+) . By Theorem 6.0.8, $\overline{ev}_-^k(\partial(\mathcal{M}/\mathbb{R}))$ does not pass through $(0, \dots, 0, \pm 1)$. Hence Claims 8.7.5 and 8.7.6 and Lemma 8.7.7 imply that there is no solution of Equation (8.7.8), a contradiction. Notice that in this argument we only need v_+ to be holomorphic near the negative end, so we only need to preglue $v_1 \cup \dots \cup v_l$ instead of glue.

(2) is similar since $\overline{ev}^k(\mathcal{M}^{\text{sing}}/\mathbb{R})$ does not intersect $(0, \dots, 0, \pm 1)$ by Theorem 6.0.7. Note that when a curve has a singular point, there is no normal bundle N ; however, the entire argument carries over to the situation of $u^*T\hat{X}^\tau$ with minimal change. \square

Let $K_1 \subset K_2 \subset \dots \subset (\mathcal{M} - \mathcal{M}^{\text{sing}})/\mathbb{R}$ be an exhaustion of $(\mathcal{M} - \mathcal{M}^{\text{sing}})/\mathbb{R}$ by compact submanifolds of dimension $k - 1$ with boundary.

Corollary 8.8.4. *For sufficiently small $\delta > 0$, sufficiently large $j > 0$, and sufficiently large $R > 0$,*

$$G|_{\mathcal{U}_{\delta, K_j}} : \mathfrak{s}^{-1}(0) \cap \mathcal{U}_{\delta, K_j} \xrightarrow{\sim} \mathcal{G}_\delta(v_-, \overline{\mathcal{M}/\mathbb{R}})$$

is a homeomorphism.

Proof. Immediate from Proposition 8.8.1 and Theorem 8.6.1. \square

8.9. Other cases. In this subsection we treat two other cases:

A. Suppose (C1)–(C4) holds with (C1) modified to (C1') so that $\gamma'' = \gamma'_+$ is positive hyperbolic and $\gamma' = \gamma_-$ is negative hyperbolic. Then Lemma 1.0.2 is modified so that σ_1 is a nonzero constant multiple of Y modulo f_{k+1}, f_{k+2}, \dots ; see Lemma 4.3.1. The rest of the discussion carries over with $(0, \dots, 0, \pm 1) \in S^{k-1}$ replaced by $(\pm 1, 0, \dots, 0) \in S^{k-1}$.

B. Consider the 3-level SFT building $v_{-1} \cup v_0 \cup v_1$ where:

- (C1'') v_1 is a cylinder from γ_+ to γ'_+ , v_0 is a cylinder from γ'_+ to γ'_- , and v_{-1} is a cylinder from γ'_- to γ_- ; we assume that γ'_+ is negative hyperbolic and γ'_- is positive hyperbolic;
- (C2'') v_0 maps to a cobordism $(\hat{X}^{\tau_0}, \hat{\alpha}^{\tau_0}, \overline{\mathcal{J}}^{\tau_0})$ for some $\tau_0 \in (0, 1)$ and v_{-1} and v_1 map to symplectizations;
- (C3'') $\text{ind}(v_{-1}) = b$, $\text{ind}(v_0) = -k$, $\text{ind}(v_1) = a$, where $a, b > 0$ and $a + b = k > 1$;
- (C4) v_0 is a k -fold unbranched cover of a transversely cut out (in a 1-parameter family) cylinder u_0 with $\text{ind}(u_0) = -1$ and v_1 is regular; and we write $v_0 = u_0 \circ \pi$, where π is the covering map.

The gluing setup is similar to the previous situation: Let

$$\mathcal{M}_1 = \mathcal{M}_{J_+}^{\text{ind}=a, \text{cyl}}(\gamma_+, \gamma'_+), \quad \mathcal{M}_{-1} = \mathcal{M}_{J_-}^{\text{ind}=b, \text{cyl}}(\gamma'_-, \gamma_-).$$

We assume that all the curves in \mathcal{M}_1 and \mathcal{M}_{-1} are immersed; modifications can be made using the analog of Proposition 8.8.1. The obstruction bundle is

$$\mathcal{O} \rightarrow [R, \infty)^{\times 2} \times (\mathcal{M}_{-1}/\mathbb{R}) \times (\mathcal{M}_1/\mathbb{R}),$$

whose fiber over (T_-, T_+, v_{-1}, v_1) is $\text{Hom}(\ker D_{v_0}^*/\mathbb{R}\langle Y \rangle, \mathbb{R})$. The pregluing is done in exactly the same way with gluing parameters T_+ and T_- , such that the pregluing between v_1 and v_0 is the same as that of Section 8.1 with parameter T_+ instead of T and the pregluing between v_{-1} and v_0 is done symmetrically about $s = 0$ with parameter T_- . Let us write η_1 for the end corresponding to $v_1(s - 2T_+, t)$ and η_{-1} for the end corresponding to $v_{-1}(s + 2T_-, t)$. Define the cut-off functions

$\beta_1(s) = \beta\left(\frac{s-T_0}{hr}\right)$, $\beta_{-1}(s) = \beta\left(\frac{-T_0-s}{hr}\right)$, and $\beta_0(s) = \beta\left(\frac{s+T_{-1}}{hr}\right)\beta\left(\frac{T_1-s}{hr}\right)$. We then define the pregluing of $v_{-1} \cup v_0 \cup v_1$ by

$$v_*(s, t) = \begin{cases} v_{1,T_1}(s, t) & \text{for } T_1 \leq s, \\ (s, t, \beta_1(s)\eta_1(s, t) + \beta_0(s)\eta_0(s, t)) & \text{for } T_0 \leq s < T_1, \\ v_0(s, t) & \text{for } -T_0 \leq s < T_0, \\ (s, t, \beta_{-1}(s)\eta_{-1}(s, t) + \beta_0(s)\eta_0(s, t)) & \text{for } -T_{-1} \leq s < -T_0, \\ v_{-1,T_{-1}}(s, t) & \text{for } s < -T_{-1}. \end{cases}$$

Assume for convenience v_{-1} , v_0 , and v_1 are all immersed. Otherwise, one can go through the proof of Proposition 8.8.1 to deal with the rest cases. Let ψ_1 , ψ_0 , and ψ_{-1} be sections of the normal bundles of η_{-1} , η_0 , and η_1 , respectively. We deform the v_* as before by

$$v = \exp_{v_*}(\beta_1\psi_1 + \beta_0\psi_0 + \beta_{-1}\psi_{-1}).$$

The obstruction section \mathfrak{s} is defined in a similar manner as in Section 8.5.

The linearized section is

$$(8.9.1) \quad \mathfrak{s}_0(T_-, T_+, v_{-1}, v_1)(\sigma) = \left\langle \tilde{\sigma}, \frac{\partial\beta_1}{\partial s}\eta_1 - \frac{\partial\beta_{-1}}{\partial s}\eta_{-1} \right\rangle,$$

where $\tilde{\sigma}$ is σ with the $f_{k+1}(t), f_{k+2}(t), \dots$ terms truncated at the positive end and $g_{-k-1}(t), g_{-k-2}(t), \dots$ truncated at the negative end. We use the basis

$$\{\sigma'_1, \dots, \sigma'_{k-1}, \sigma_k\}$$

from Lemma 4.3.2 to analyze $\mathfrak{s}_0^{-1}(0)$. By Theorems 6.0.4 and 6.0.7 and our genericity assumption, there exist evaluation maps

$$\tilde{e}v_-^a(\gamma_+, \gamma'_+) : \mathcal{M}_1/\mathbb{R} \rightarrow S^{a-1},$$

$$\tilde{e}v_+^b(\gamma'_-, \gamma_-) : \mathcal{M}_{-1}/\mathbb{R} \rightarrow S^{b-1}.$$

The evaluation maps will be abbreviated $\tilde{e}v(\eta_1)$ and $\tilde{e}v(\eta_{-1})$ when the ends are understood.

Lemma 8.9.1. *Let $K \subset (\mathcal{M}_{-1}/\mathbb{R}) \times (\mathcal{M}_1/\mathbb{R})$ be a compact $(k-2)$ -dimensional submanifold with boundary. Then for $R \gg 0$, there are no zeros of \mathfrak{s}_0 on $[R, \infty)^{\times 2} \times \partial K$.*

Proof. Suppose $R \gg 0$. Let $(v_{-1}, v_1) \in K$ and let us denote their translated ends by

$$\eta_1(s, t) = \sum_{i=1}^{\infty} c_i e^{\lambda_i(s-2T_+)} f_i(t), \quad \eta_{-1}(s, t) = \sum_{i=-\infty}^{-1} d_i e^{\lambda'_i(s+2T_-)} g_i(t).$$

Suppose that $\lambda_a < \lambda_{a+1}$. Let us write

$$\tilde{\sigma}'_i(s, t) = \begin{cases} \sum_{i \leq j \leq k} c_{i,j} e^{-\lambda_j s} f_j(t), & c_{i,i} = 1, \\ \sum_{-k \leq j \leq -k+i-1} d_{i,j} e^{-\lambda'_j s} g_j(t), & d_{i,-k+i-1} \neq 0, \end{cases}$$

at the positive and negative ends. By a computation similar to that of Lemma 8.7.1,

$$(8.9.2) \quad \left\langle \tilde{\sigma}'_i, \frac{\partial \beta_1}{\partial s} \eta_1 \right\rangle = \sum_{i \leq j \leq k} c_{i,j} c_j e^{-2\lambda_j T_+},$$

$$(8.9.3) \quad \left\langle \tilde{\sigma}'_i, \frac{\partial \beta_{-1}}{\partial s} \eta_{-1} \right\rangle = \sum_{-k \leq j \leq -k+i-1} d_{i,j} d_j e^{2\lambda'_j T_-}.$$

Now let us write

$$c_{i,*} = (c_{i,1}, \dots, c_{i,k}), \quad d_{i,*} = (d_{i,-k}, \dots, d_{i,-1}),$$

where we are setting $c_{i,1} = \dots = c_{i,i-1} = 0$ and $d_{i,-k+i} = \dots = d_{i,-1} = 0$, and

$$C_1 = (c_1 e^{-2\lambda_1 T_+}, \dots, c_a e^{-2\lambda_a T_+}), \quad C_2 = (c_{a+1} e^{-2\lambda_{a+1} T_+}, \dots, c_k e^{-2\lambda_k T_+}),$$

$$D_{-2} = (d_{-k} e^{2\lambda'_{-k} T_-}, \dots, d_{-k+a-1} e^{2\lambda'_{-k+a-1} T_-}),$$

$$D_{-1} = (d_{-b} e^{2\lambda'_{-b} T_-}, \dots, d_{-1} e^{2\lambda'_{-1} T_-}).$$

Recall that $a + b = k$. Since $\lambda_a < \lambda_{a+1}$ and (c_1, \dots, c_a) and (d_{-b}, \dots, d_{-1}) are nonzero, it follows that $|C_2| \ll |C_1|$ and $|D_{-2}| \ll |D_{-1}|$ for $R \gg 0$.

Suppose $i = j_0$ maximizes (8.9.2), where we are ranging over $i = 1, \dots, a$. We claim that if $\mathfrak{s}_0(\sigma'_{j_0}) = 0$, then $|C_1| \leq c|D_{-2}|$ for some constant $c > 0$ which does not depend on (T_-, T_+, v_{-1}, v_1) , provided $R \gg 0$. First observe that $\{c_{i,*}\}_{i=1}^k$ and $\{d_{i,*}\}_{i=1}^k$ form bases of \mathbb{R}^k and that (8.9.2) and (8.9.3) can be written as:

$$\langle c_{i,*}, (C_1, C_2) \rangle \quad \text{and} \quad \langle d_{i,*}, (D_{-2}, D_{-1}) \rangle,$$

where we are using the standard inner product on \mathbb{R}^k . It is not hard to see that there exist a constant $c > 0$, which only depends on $\{c_{i,*}\}_{i=1}^k$, and an integer $i = j'_0$, which depends on (C_1, C_2) , such that

$$|\langle c_{j'_0,*}, (C_1, C_2) \rangle| \geq c \cdot |c_{j'_0,*}| \cdot |(C_1, C_2)|.$$

(Roughly speaking, some vector $c_{j'_0,*}$ makes an angle with (C_1, C_2) which is bounded away from $\frac{\pi}{2}$.) If $i = j_0$ maximizes (8.9.2) and $\mathfrak{s}_0(\sigma'_{j_0}) = 0$, then

$$c \cdot |c_{j'_0,*}| \cdot |(C_1, C_2)| \leq |\langle c_{j_0,*}, (C_1, C_2) \rangle| \leq |\langle d_{j_0,*}, (D_{-2}, D_{-1}) \rangle|.$$

Hence, by Lemma 4.3.2 and the fact that $|D_{-2}| \ll |D_{-1}|$,

$$c \cdot |c_{j'_0,*}| \cdot |C_1| \leq |d_{j_0,*}| \cdot |D_{-2}|,$$

which implies the claim.

Combining the inequalities $|C_2| \ll |C_1|$, $|C_1| \leq c|D_{-2}|$, and $|D_{-2}| \ll |D_{-1}|$, we obtain $|C_2| \ll |D_{-1}|$. On the other hand, applying the above argument to $\tilde{\sigma}'_{j_1}$, $a + 1 \leq j_1 < k$, such that $\mathfrak{s}_0(\sigma'_{j_1}) = 0$, we obtain $|D_{-1}| \leq c|C_2|$, a contradiction. This proves the lemma when $\lambda_a < \lambda_{a+1}$.

When $\lambda_a = \lambda_{a+1}$, the only case that is not treated by the above argument is when

$$\begin{aligned}\eta_1(s, t) &= (c_a f_a(t) + c_{a+1} f_{a+1}(t)) e^{\lambda_a(s-2T_+)}, \\ \eta_{-1}(s, t) &= (d_{-b-1} g_{-b-1}(t) + d_{-b} g_{-b}(t)) e^{\lambda'_{-b-1}(s+2T_-)},\end{aligned}$$

up to a small error. By the transversality of the evaluation maps, there is only a finite number of possible values for (c_a, c_{a+1}) and (d_{-b-1}, d_{-b}) ; moreover their genericity implies that, for each (T_-, T_+, v_{-1}, v_1) , there exists (r_a, r_{a+1}) such that $\mathfrak{s}_0(T_-, T_+, v_{-1}, v_1)(r_a \sigma'_a + r_{a+1} \sigma'_{a+1}) \neq 0$. This proves the lemma. \square

The existence of a homotopy \mathfrak{s}_ζ from $\mathfrak{s}_1 = \mathfrak{s}$ to \mathfrak{s}_0 without zeros on $[R, \infty)^{\times 2} \times \partial K$ is proved as in Proposition 8.7.2, using Lemma 8.9.1. The higher codimension strata are eliminated using the argument of Section 8.8.

9. ORIENTATIONS

In this section we discuss orientations for the moduli spaces. Section 9.1 and Sections 9.2.1–9.2.2 are standard.

9.1. $\partial^2 = 0$.

9.1.1. Signs in the definition of ∂ . We first discuss orientations involved in the definition of ∂ . Let $u \in \mathcal{M}_J^{\text{ind}=\ell, \text{cyl}}(\gamma, \gamma')$, where we are suppressing asymptotic markers from the notation. For simplicity let us assume that u is immersed. Let $D = D_u$ be the linearized normal $\bar{\partial}$ -operator and let

$$\det(D) = \Lambda^{\text{top}} \ker D \otimes \Lambda^{\text{top}}(\text{coker } D)^*$$

be its determinant line. We define an equivalence relation \sim on $\det(D) - \{0\}$ as follows: $\xi_1 \sim \xi_2$ if $\xi_1 = c\xi_2$ for $c \in \mathbb{R}^+$. The equivalence class of $\xi \in \det(D) - \{0\}$ is denoted by $[\xi]$. Let $\mathfrak{o}(D)$ be the orientation of $\det(D)$ given by [BM]. An orientation of $\det(D)$ can be viewed as an equivalence class of $\det(D) - \{0\}$.

When $\text{ind}(u) = 1$, we can write $\mathfrak{o}(D) = [\text{sgn}(u) \cdot \partial_s]$, where ∂_s refers to the section of the normal bundle corresponding to the infinitesimal translation in the s -direction and $\text{sgn}(u) = \pm 1$.

Sign assignment. In the definition of ∂ , we assign the sign $\text{sgn}(u)$ to $[u] \in \mathcal{M}_J^{\text{ind}=1, \text{cyl}}(\gamma, \gamma')/\mathbb{R}$.

9.1.2. $\partial^2 = 0$. Next we discuss orientations in the proof of $\partial^2 = 0$. We are gluing/pregluing $u_1 \in \mathcal{M}_J^{\text{ind}=1, \text{cyl}}(\gamma'', \gamma')$ and $u_2 \in \mathcal{M}_J^{\text{ind}=1, \text{cyl}}(\gamma, \gamma'')$. Assume u_1 and u_2 are regular. Let $u_1 \# u_2$ be a pregluing of u_1 and u_2 and let $D_{u_1 \# u_2}$ be the linearized normal $\bar{\partial}$ -operator for $u_1 \# u_2$. We assume that the “neck length” T is sufficiently large. Then by the gluing property for coherent orientations from [BM], there is an isomorphism

$$\det(D_{u_1 \# u_2}) \simeq \det(D_{u_1}) \otimes \det(D_{u_2})$$

which is natural up to a positive constant and

$$(9.1.1) \quad \mathfrak{o}(D_{u_1 \# u_2}) = \mathfrak{o}(D_{u_1}) \widetilde{\otimes} \mathfrak{o}(D_{u_2}),$$

where the right-hand side is defined as follows: We use cutoff functions $\beta_i : \mathbb{R} \rightarrow [0, 1]$, $i = 1, 2$, that are analogous to β_{\pm} from Section 8.1. Given $\xi_i \in \ker D_{u_i}$, we translate by $\pm T$ and damp it out at the positive or negative end (as appropriate) by multiplying by β_i . This yields $\xi'_i = \beta_i(\xi_i)_T$. Here the subscript T indicates a translation *which depends on T* . We then view ξ'_i as an element of the domain of $D_{u_1 \# u_2}$ and take the L^2 -orthogonal projection to $\ker D_{u_1 \# u_2}$ to obtain $\widehat{\xi}_i$. Finally, if

$$\mathfrak{o}(D_{u_1}) = [\operatorname{sgn}(u_1) \partial_s^1] \quad \text{and} \quad \mathfrak{o}(D_{u_2}) = [\operatorname{sgn}(u_2) \partial_s^2],$$

where ∂_s^i , $i = 1, 2$, is the vector field that corresponds to translation in s -direction inside the moduli space corresponding to u_i , then

$$\mathfrak{o}(D_{u_1}) \widetilde{\otimes} \mathfrak{o}(D_{u_2}) := [\operatorname{sgn}(u_1) \widehat{\partial}_s^1 \wedge \operatorname{sgn}(u_2) \widehat{\partial}_s^2].$$

Now let $u_1 \cup u_2$ and $u_1^b \cup u_2^b$ be the two boundary points of a component \mathcal{N} of $\overline{\mathcal{M}}/\mathbb{R}$. Let a be a nonsingular vector field of \mathcal{N} that points away from $u_1 \cup u_2$ and towards $u_1^b \cup u_2^b$, and let ∂_s^{12} be the vector field of \mathcal{M} that corresponds to translation in the s -direction. If we denote $\mathfrak{o}(D_{u_1 \# u_2}) = [\operatorname{sgn}(a) \partial_s^{12} \wedge a]$, then $\widehat{\partial}_s^1 + \widehat{\partial}_s^2$ corresponds to ∂_s^{12} and $\widehat{\partial}_s^1 - \widehat{\partial}_s^2$ corresponds to a . Therefore,

$$[\operatorname{sgn}(u_1) \widehat{\partial}_s^1 \wedge \operatorname{sgn}(u_2) \widehat{\partial}_s^2] = [\operatorname{sgn}(a) (\widehat{\partial}_s^1 + \widehat{\partial}_s^2) \wedge (\widehat{\partial}_s^1 - \widehat{\partial}_s^2)],$$

and $\operatorname{sgn}(u_1) \operatorname{sgn}(u_2) = -\operatorname{sgn}(a)$. The analogous computation for $u_1^b \cup u_2^b$ implies

$$(9.1.2) \quad \operatorname{sgn}(u_1^b) \operatorname{sgn}(u_2^b) = \operatorname{sgn}(a) = -\operatorname{sgn}(u_1) \operatorname{sgn}(u_2).$$

This is the desired relation for $\partial^2 = 0$.

9.2. Chain homotopy.

9.2.1. *A lemma.* In this subsection we will make frequent use of the following standard lemma (cf. [FO3, p. 676], for example):

Lemma 9.2.1. *If $\phi : V \rightarrow W$ is a Fredholm map and E is a finite-dimensional subspace of W such that $W = \operatorname{Im} \phi + E$, then there is an isomorphism*

$$\Phi_E : \det \phi \xrightarrow{\sim} \det \phi^{-1}(E) \otimes \det E^*,$$

which is natural up to a positive constant.

More explicitly, if $\phi^{-1}(E) = \ker \phi \oplus F$ and $E = \operatorname{coker} \phi \oplus \phi(F)$, and $\ker \phi$, F , $\operatorname{coker} \phi$, and $\phi(F)$ have bases $\{v_1, \dots, v_m\}$, $\{f_1, \dots, f_\ell\}$, $\{w_1, \dots, w_n\}$, and $\{\phi(f_1), \dots, \phi(f_\ell)\}$, then the isomorphism Φ_E is given by:

$$[v_1 \wedge \dots \wedge v_m \otimes w_n^* \wedge \dots \wedge w_1^*] \mapsto [v_1 \wedge \dots \wedge v_m \wedge f_1 \wedge \dots \wedge f_\ell \otimes \phi(f_\ell)^* \wedge \dots \wedge \phi(f_1)^* \wedge w_n^* \wedge \dots \wedge w_1^*],$$

where $\{w_1^*, \dots, w_n^*\}$ is the dual basis to $\{w_1, \dots, w_n\}$ and $\{\phi(f_1)^*, \dots, \phi(f_\ell)^*\}$ is the dual basis to $\{\phi(f_1), \dots, \phi(f_\ell)\}$.

9.2.2. *The $k = 1$ case.* We mostly use the notation from Section 7.1. Let

$$\mathcal{M}^0 = \coprod_{0 \leq \tau \leq 1} \mathcal{M}_{\mathcal{J}^\tau}^{\text{ind}=0, \text{cyl}}(\gamma_+, \gamma_-)$$

and let $\pi : \mathcal{M}^0 \rightarrow [0, 1]$ be the projection to $\tau \in [0, 1]$.

Consider

$$v_0 \cup v_1 \in \left(\mathcal{M}_{\mathcal{J}^{\tau_l}}^{\text{ind}=-k, \text{cyl}}(\gamma'_+, \gamma_-) \times \mathcal{M}_{J_+}^{\text{ind}=k, \text{cyl}}(\gamma_+, \gamma'_+) / \mathbb{R} \right) / \sim.$$

Let

$$D_{v_0} : \mathcal{H}_1(\dot{F}_0, N_{v_0}) \rightarrow \mathcal{H}_0(\dot{F}_0, \Lambda^{0,1} T^* \dot{F}_0 \otimes N_{v_0}),$$

$$D_{v_1} : \mathcal{H}_1(\dot{F}_1, N_{v_1}) \rightarrow \mathcal{H}_0(\dot{F}_1, \Lambda^{0,1} T^* \dot{F}_1 \otimes N_{v_1}),$$

be the linearized normal $\bar{\partial}$ -operators for $v_0 : \dot{F}_0 \rightarrow \widehat{X}^{\tau_0}$ and $v_1 : \dot{F}_1 \rightarrow \mathbb{R} \times M_+$, where \mathcal{H}_0 and \mathcal{H}_1 are the Morrey spaces described in Section 8.3. We then write

$$\begin{aligned} \tilde{D}_{v_0} : \mathcal{H}_1(\dot{F}_0, N_{v_0}) \oplus \mathbb{R} &\rightarrow \mathcal{H}_0(\dot{F}_0, \Lambda^{0,1} T^* \dot{F}_0 \otimes N_{v_0}), \\ (\xi, c) &\mapsto D_{v_0} \xi + c Y', \end{aligned}$$

where the generator of the summand \mathbb{R} is denoted by a .

We first consider the case $k = 1$. We define $\text{sgn}(v_0)$ and $\text{sgn}(v_1)$ by

$$\mathfrak{o}(D_{v_0}) = [\text{sgn}(v_0)(Y')^*], \quad \mathfrak{o}(D_{v_1}) = [\text{sgn}(v_1)\partial_s].$$

Then, by Lemma 9.2.1,

$$\mathfrak{o}(\tilde{D}_{v_0}) = [\text{sgn}(v_0)a \otimes (Y')^*] = \text{sgn}(v_0)[1],$$

since \tilde{D}_{v_0} maps $a \mapsto Y'$.

Let $\tilde{D} = \tilde{D}_{v_0 \# v_1}$ be the operator obtained from pregluing \tilde{D}_{v_0} and D_{v_1} :

$$\begin{aligned} \tilde{D} : \mathcal{H}_1(\dot{F}_0 \# \dot{F}_1, N_{v_0 \# v_1}) \oplus \mathbb{R} &\rightarrow \mathcal{H}_0(\dot{F}_0 \# \dot{F}_1, \Lambda^{0,1} T^*(\dot{F}_0 \# \dot{F}_1) \otimes N_{v_0 \# v_1}), \\ (\xi, c) &\mapsto D_{v_0 \# v_1} \xi + c \beta_0 Y', \end{aligned}$$

where β_0, β_1 are the same as the cutoff functions β_-, β_+ from Section 8.1. For $i = 0, 1$ we also have maps

$$\begin{aligned} \mathcal{H}_1(\dot{F}_i, N_{v_i}) &\rightarrow \mathcal{H}_1(\dot{F}_0 \# \dot{F}_1, N_{v_0 \# v_1}), \\ \mathcal{H}_0(\dot{F}_i, \Lambda^{0,1} T^* \dot{F}_i \otimes N_{v_i}) &\rightarrow \mathcal{H}_0(\dot{F}_0 \# \dot{F}_1, \Lambda^{0,1} T^*(\dot{F}_0 \# \dot{F}_1) \otimes N_{v_0 \# v_1}), \\ \xi_0 &\mapsto \xi'_0 := \beta_0 \xi_0, \quad \xi_1 \mapsto \xi'_1 := \beta_1(\xi_1)_T \end{aligned}$$

which restrict to inclusions on $\ker D_{v_1}$ and $\text{coker } D_{v_0}$. Here the subscript T indicates a translation which depends on T and ξ'_i is viewed as a section of the appropriate bundle. For $\xi_i \in \mathcal{H}_1(\dot{F}_i, N_{v_i})$, we let $\tilde{\xi}_i$ be the L^2 -orthogonal projection of ξ'_i to $\ker \tilde{D}$. By abuse of notation, let ∂_τ be the pullback of ∂_τ under the projection $\pi : \mathcal{M}^0 \rightarrow [0, 1]$. If $v_0 \cup v_1$ is on $\partial \mathcal{M}^0$ such that ∂_τ points away from (resp. towards) $v_0 \cup v_1$ and towards (resp. away from) the interior of \mathcal{M}^0 , then ∂_τ corresponds to $-\hat{\partial}_s$ (resp. $\hat{\partial}_s$). By the gluing property for coherent orientations we have

$$\mathfrak{o}(\tilde{D}) = \text{sgn}(v_0) \text{sgn}(v_1)[\hat{\partial}_s] = \text{sgn}(v_0) \text{sgn}(v_1)[-\partial_\tau]$$

(resp. $\mathfrak{o}(\tilde{D}) = \text{sgn}(v_0) \text{sgn}(v_1)[\partial_\tau]$) and the sign $\text{sgn}(v_0) \text{sgn}(v_1)$ assigned to $v_0 \cup v_1$ agrees with the boundary orientation of \mathcal{M}^0 .

9.2.3. *The case $k > 1$.* Next we consider the case $k > 1$. Define $\text{sgn}(v_0)$ and $\text{sgn}(v_1)$ by

$$\mathfrak{o}(D_{v_0}) = [\text{sgn}(v_0) \sigma_k^* \wedge \cdots \wedge \sigma_1^*], \quad \mathfrak{o}(D_{v_1}) = [\text{sgn}(v_1) e_1 \wedge \cdots \wedge e_k],$$

where $e_i \in \ker D_{v_1}$, $1 \leq i \leq k$, corresponds to f_i . Then

$$\mathfrak{o}(\tilde{D}_{v_0}) = [\text{sgn}(v_0) a \otimes \sigma_k^* \wedge \cdots \wedge \sigma_1^*] = [\text{sgn}(v_0) \sigma_{k-1}^* \wedge \cdots \wedge \sigma_1^*].$$

Notice that $\text{sgn}(v_1)$ is locally constant.

We will now define $\mathfrak{o}(\tilde{D}_{v_0}) \tilde{\otimes} \mathfrak{o}(D_{v_1})$. Let

$$F = \mathbb{R}\langle e'_1, \dots, e'_k \rangle, \quad E = \mathbb{R}\langle \sigma'_1, \dots, \sigma'_k \rangle.$$

We are assuming that $T \gg 0$ so that $D_{v_0 \# v_1}$ is an isomorphism and the composition of $\tilde{D}|_F$ and the L^2 -projection $p : \tilde{D}(F) \rightarrow E$ is also an isomorphism. Then we set $\hat{\xi} = \xi' - D_{v_0 \# v_1}^{-1}((1-p)\tilde{D}(\xi'))$ and

$$(9.2.1) \quad \mathfrak{o}(\tilde{D}_{v_0}) \tilde{\otimes} \mathfrak{o}(D_{v_1}) := \text{sgn}(v_0) \text{sgn}(v_1) [\hat{e}_1 \wedge \cdots \wedge \hat{e}_k \otimes (\sigma'_{k-1})^* \wedge \cdots \wedge (\sigma'_1)^*],$$

which makes sense in light of Lemma 9.2.1.

The gluing property for coherent orientations (in a slightly more general form than that of [BM]) implies:

Lemma 9.2.2. *For $T \gg 0$, $\mathfrak{o}(\tilde{D}) = \mathfrak{o}(\tilde{D}_{v_0}) \tilde{\otimes} \mathfrak{o}(D_{v_1})$. Hence*

$$\mathfrak{o}(\tilde{D}) = \text{sgn}(v_0) \text{sgn}(v_1) [\hat{e}_1 \wedge \cdots \wedge \hat{e}_k \otimes (\sigma'_{k-1})^* \wedge \cdots \wedge (\sigma'_1)^*].$$

Let \tilde{D}_v be the operator for the glued curve v , given as follows:

$$\begin{aligned} \tilde{D}_v : \mathcal{H}_1(\dot{F}_0 \# \dot{F}_1, N_v) \oplus \mathbb{R} &\rightarrow \mathcal{H}_0(\dot{F}_0 \# \dot{F}_1, \Lambda^{0,1} T^*(\dot{F}_0 \# \dot{F}_1) \otimes N_v), \\ (\xi, c) &\mapsto D_v \xi + c \beta_0 Y', \end{aligned}$$

where D_v is the linearized normal $\bar{\partial}$ -operator for v . Then the analog of Lemma 9.2.2 also holds for \tilde{D}_v .

9.2.4. *Comparison with signs of zeros of $d\mathfrak{s}$.* Next we compare $\mathfrak{o}(\tilde{D}_v)$ with the signs of zeros of $d\mathfrak{s}$. As before we write $\mathcal{M} = \mathcal{M}_{J_+}^{\text{ind}=k, \text{cyl}}(\gamma_+, \gamma'_+)$. Assume that $T \gg 0$ is sufficiently generic so that $\mathfrak{s} : \{T\} \times \mathcal{M}/\mathbb{R} \rightarrow \mathcal{O}$ is transverse to the zero section. Let $v_1 \in \mathfrak{s}^{-1}(0)$. In Section 9.2.4 only, we write $v_+ = v_1$ so that we have agreement with Section 8. We orient $T_{v_+}(\mathcal{M}/\mathbb{R})$ by

$$-[e_1 \wedge \cdots \wedge e_k \otimes \partial_s^*] = -[e_1^\# \wedge \cdots \wedge e_{k-1}^\# \wedge \partial_r \otimes \partial_s^*] = [e_1^\# \wedge \cdots \wedge e_{k-1}^\#],$$

where ∂_r is the outward radial vector field on \mathbb{R}^k and $e_1^\#, \dots, e_{k-1}^\#$ are tangent to $S^{k-1} \subset \mathbb{R}^k$ at $\tilde{e}^k(v_+)$ so that $[e_1^\# \wedge \cdots \wedge e_{k-1}^\# \wedge \partial_r] = [e_1 \wedge \cdots \wedge e_k]$, and orient the fiber \mathcal{O}_{T, v_+} by $[\sigma_{k-1}^* \wedge \cdots \wedge \sigma_1^*]$.

Sign of $d\mathfrak{s}$. We define $\text{sgn } d\mathfrak{s}(v_+) \in \{\pm 1\}$ as the sign of $\det(\pi_{\mathcal{O}} \circ d\mathfrak{s}(v_+))$, where $\pi_{\mathcal{O}}$ is the projection to the fiber \mathcal{O}_{T, v_+} .

Let

$$F^\sharp = \mathbb{R}\langle (e_1^\sharp)', \dots, (e_{k-1}^\sharp)' \rangle, \quad E^\sharp = \mathbb{R}\langle \sigma_1', \dots, \sigma_{k-1}' \rangle,$$

and let $D_v^\sharp : F^\sharp \rightarrow E^\sharp$ be the composition of $\tilde{D}_v|_{F^\sharp}$ and the L^2 -projection to E^\sharp . Also we orient F^\sharp by $[(e_1^\sharp)' \wedge \dots \wedge (e_{k-1}^\sharp)']$ and E^\sharp by $[\sigma_1' \wedge \dots \wedge \sigma_{k-1}']$; these are analogous to the orientations for $T_{v_+}(\mathcal{M}/\mathbb{R})$ and \mathcal{O}_{T, v_+} .

Lemma 9.2.3. $\text{sgn } d\mathfrak{s}(v_+) = \text{sgn } \det D_v^\sharp$.

Proof. Using the notation from Section 8, given

$$v = \exp_{v_*}(\beta_+ \psi_+ + \beta_- \psi_-),$$

we consider $\bar{\partial}_{\mathcal{J}^\tau} v$, which equals the left-hand side of Equation (8.2.2). Here $\beta_0 = \beta_-$ and $\beta_1 = \beta_+$. For a zero v_+ of \mathfrak{s} , there exist (ψ_+, ψ_-, τ) so that $\bar{\partial}_{\mathcal{J}^\tau} v = 0$; in other words, (ψ_+, ψ_-, τ) solves Equations (8.2.6)–(8.2.8).

Consider the variation $\beta_+ \phi_+$ with $\phi_+ \in \ker D_+$. We claim that

$$(9.2.2) \quad d\mathfrak{s}(v_+)(\phi_+)(\sigma) = \left\langle \sigma, \tilde{D}_v(\beta_+(\phi_+ + \psi_+^b) + \beta_- \psi_-^b, \tau^b) \right\rangle,$$

where $(\psi_+ + \phi_+ + \psi_+^b, \psi_- + \psi_-^b, \tau + \tau^b)$ satisfy Equations (8.2.6) and (8.2.7), with (ψ_+, ψ_-, τ) replaced by $(\psi_+ + \phi_+ + \psi_+^b, \psi_- + \psi_-^b, \tau + \tau^b)$, and $\sigma \in \ker D_-^*/\mathbb{R}\langle Y \rangle$. Although strictly speaking not necessary, we write out the left-hand side of Equation (8.2.2) for

$$v_{\tilde{\phi}_+} = \exp_{v_*}(\beta_+(\psi_+ + \phi_+ + \psi_+^b) + \beta_-(\psi_- + \psi_-^b))$$

and $\tau + \tau^b$, using the fact that $\bar{\partial}_{\mathcal{J}^\tau} v = 0$:

$$\begin{aligned} & \beta_- \left(D_- \psi_-^b + \tau^b Y' + \frac{\partial \beta_+}{\partial s}(\phi_+ + \psi_+^b) + \Delta \mathcal{R}'_-(\psi_- + \psi_-^b, \tau - \tau_0 + \tau^b) \right) \\ & + \beta_+ \left(D_+(\phi_+ + \psi_+^b) + \frac{\partial \beta_-}{\partial s}(\psi_-^b) + \Delta \mathcal{R}'_+(\psi_+ + \phi_+ + \psi_+^b) \right), \end{aligned}$$

up to first order in $\phi_+, \psi_+^b, \psi_-^b, \tau^b$. Here $\Delta \mathcal{R}'_+$ has terms of the form $B(\psi_-, \psi_-^b)$, $B(\psi_-, \tau^b)$, $B(\tau - \tau_0, \psi_-^b)$, and $B(\tau - \tau_0, \tau^b)$; and $\Delta \mathcal{R}'_-$ has terms of the form $B(\psi_+, \phi_+ + \psi_+^b)$. By $B(r_0, r_1)$ we mean a term that is linear in r_1 with coefficients that are functions of r_0 . Then, by Equations (8.2.6) and (8.2.7), $d\mathfrak{s}(v_+)(\phi_+)$ is given by

$$(9.2.3) \quad \sigma \mapsto \left\langle \sigma, \frac{\partial \beta_+}{\partial s}(\phi_+ + \psi_+^b) + \Delta \mathcal{R}'_-(\psi_- + \psi_-^b, \tau - \tau_0 + \tau^b) \right\rangle,$$

whereas $\langle \sigma, \tilde{D}_v(\beta_+(\phi_+ + \psi_+^b) + \beta_- \psi_-^b, \tau^b) \rangle$ is given by

$$\begin{aligned} (9.2.4) \quad \sigma \mapsto & \left\langle \sigma, \beta_- \left(\frac{\partial \beta_+}{\partial s}(\phi_+ + \psi_+^b) + \Delta \mathcal{R}'_-(\psi_- + \psi_-^b, \tau - \tau_0 + \tau^b) \right) \right\rangle \\ & = \left\langle \sigma, \frac{\partial \beta_+}{\partial s}(\phi_+ + \psi_+^b) + \Delta \mathcal{R}'_-(\psi_- + \psi_-^b, \tau - \tau_0 + \tau^b) \right\rangle, \end{aligned}$$

since $\beta_- = 1$ on the support of $\frac{\partial \beta_+}{\partial s}$ and $\Delta \mathcal{R}'_-$. This proves the claim.

The claim, together with Lemma 9.2.2 for \tilde{D}_v , implies the lemma. \square

9.2.5. *From $d\mathfrak{s}$ to $d\mathfrak{s}_0$.* The homotopy \mathfrak{s}_ζ , $\zeta \in [0, 1]$, from $\mathfrak{s} = \mathfrak{s}_1$ to \mathfrak{s}_0 gives an oriented cobordism from $\mathfrak{s}^{-1}(0)$ to $\mathfrak{s}_0^{-1}(0)$. In view of Lemmas 9.2.2 and 9.2.3 and the facts that $\text{sgn}(v_0)$ is fixed and $\text{sgn}(v_1)$ is locally constant, it suffices to keep track of $\text{sgn } d\mathfrak{s}_\zeta$ as we go from \mathfrak{s} to \mathfrak{s}_0 .

Let $v_1 \in \mathfrak{s}_0^{-1}(0)$. We will determine $\text{sgn } d\mathfrak{s}_0(v_1)$, using Equation (8.7.3). There are two cases: $v_1 = v_1^+$ or v_1^- , where $\tilde{e}v_-^k(v_1^\pm) = (0, \dots, 0, \pm 1)$.

Lemma 9.2.4. $\text{sgn } d\mathfrak{s}_0(v_1^\pm) = \pm 1$.

Proof. The tangent space $T_{v_1^\pm}(\mathcal{M}/\mathbb{R})$ is oriented by $\pm[e_1 \wedge \dots \wedge e_{k-1}]$. For $i = 1, \dots, k-1$, $d\mathfrak{s}_0(v_1^+)$ maps e_i to some positive multiple of σ_i^* by Equation (8.7.3). Hence $\text{sgn } d\mathfrak{s}_0(v_1^+) = +1$. Similarly, $\text{sgn } d\mathfrak{s}_0(v_1^-) = -1$. \square

Let $\tilde{D}_{v_0 \# v_1^\pm}$ be the analog of \tilde{D}_v for $v_0 \# v_1^\pm$, with one modification: $D_{v_0 \# v_1^\pm}$ is the linearization of $\bar{\partial}_{\mathcal{T}^\tau}(v_0 \# v_1^\pm)$, where the term $F_-(\psi_-, \tau - \tau_0)$ of \mathcal{R}_- is set to zero. This is analogous to changing \mathfrak{s} to \mathfrak{s}_0 .

By Lemmas 9.2.2 and 9.2.3 adapted to $\tilde{D}_{v_0 \# v_1^\pm}$ and Lemma 9.2.4,

(9.2.5)

$$\begin{aligned} \mathfrak{o}(\tilde{D}_{v_0 \# v_1^\pm}) &= \text{sgn}(v_0) \text{sgn}(v_1^\pm) [\hat{e}_1 \wedge \dots \wedge \hat{e}_k \otimes (\sigma'_{k-1})^* \wedge \dots \wedge (\sigma'_1)^*] \\ &= (-1)^{k-1} \text{sgn}(v_0) \text{sgn}(v_1^\pm) [\pm \partial_r \wedge \hat{e}_1 \wedge \dots \wedge \hat{e}_{k-1} \otimes (\sigma'_{k-1})^* \wedge \dots \wedge (\sigma'_1)^*] \\ &= (-1)^{k-1} \text{sgn}(v_0) \text{sgn}(v_1^\pm) (\pm 1) \text{sgn } d\mathfrak{s}_0(v_1^\pm) [\pm \partial_r] \\ &= (-1)^k \text{sgn}(v_0) \text{sgn}(v_1^\pm) [\mp \partial_r] \end{aligned}$$

where ∂_r is the radial vector field for \mathbb{R}^k . Note that $e_k = \partial_r$ at $(0, \dots, 0, 1)$ and $e_k = -\partial_r$ at $(0, \dots, 0, -1)$.

9.2.6. *Orientations over ν .* Recall that the embedded arc $\nu \subset S^{k-1}$ is oriented from $(0, \dots, 0, 1)$ to $(0, \dots, 0, -1)$. Let $\tilde{\nu}$ be a connected component of $(\tilde{e}v_-^k)^{-1}(\nu)$.

Suppose v_1^\pm are the endpoints of $\tilde{\nu}$ over $(0, \dots, 0, \pm 1)$. We can view $\dot{\nu}$ as the continuation of the tangent vector field $-\partial_r$ at v_1^+ and ∂_r as the continuation of $\dot{\nu}$ at v_1^- . Since $\text{sgn}(v_0)$ is constant and $\text{sgn}(v_1^+) = \text{sgn}(v_1^-)$, the orientations of $v_0 \cup v_1^+$ and $v_0 \cup v_1^-$ that come from Equation (9.2.5) are consistent with the boundary orientation of $v_0 \cup \partial\tilde{\nu}$ (up to an overall sign), i.e., the signs are opposite.

Next consider

$$w_1 \cup w_2 \in \left(\mathcal{M}_+^{k-1}(\zeta_+, \gamma'_+)/\mathbb{R} \times \mathcal{M}_{J_+}^{\text{ind}=1, \text{cyl}}(\gamma_+, \zeta_+)/\mathbb{R} \right) / \sim,$$

i.e., $w_1 \cup w_2$ is a boundary point of $\tilde{\nu}$ that lies over the interior of ν . We define $\text{sgn}(w_1)$ by

$$\mathfrak{o}(D_{w_1}) = [\text{sgn}(w_1) \mathfrak{o}(N) \wedge \partial_s^1],$$

where N is the normal bundle to ν inside S^{k-1} and $\mathfrak{o}(N)$ is (a representative of) the orientation for N such that $[\mathfrak{o}(N) \wedge \dot{\nu} \wedge \partial_r] = [e_1 \wedge \dots \wedge e_k]$. Also define

$\text{sgn}(w_2)$ by

$$\mathfrak{o}(D_{w_2}) = [\text{sgn}(w_2)\partial_s^2].$$

Then

$$\mathfrak{o}(D_{w_1\#w_2}) = \mathfrak{o}(D_{w_1}) \widetilde{\otimes} \mathfrak{o}(D_{w_2}) = \text{sgn}(w_1) \text{sgn}(w_2) [\widehat{\mathfrak{o}(N)} \wedge \widehat{\partial}_s^1 \wedge \widehat{\partial}_s^2],$$

where $\widetilde{\otimes}$ is defined as in the discussion after Equation (9.1.1).

If $\dot{\nu}$ points from the interior of $\tilde{\nu}$ towards $w_1 \cup w_2$, then $\widehat{\partial}_s^1 + \widehat{\partial}_s^2$ corresponds to ∂_s^{12} or $-\partial_r$, and $-\widehat{\partial}_s^1 + \widehat{\partial}_s^2$ corresponds to $\dot{\nu}$. Hence

$$\begin{aligned} (9.2.6) \quad \mathfrak{o}(D_{w_1\#w_2}) &= -\text{sgn}(w_1) \text{sgn}(w_2) [\widehat{\mathfrak{o}(N)} \wedge \dot{\nu} \wedge \partial_s^{12}] \\ &= -\text{sgn}(w_1) \text{sgn}(w_2) [\widehat{\mathfrak{o}(N)} \wedge \dot{\nu} \wedge -\partial_r] \\ &= \text{sgn}(w_1) \text{sgn}(w_2) [e_1 \wedge \cdots \wedge e_k]. \end{aligned}$$

In particular, $\text{sgn}(w_1) \text{sgn}(w_2) = \text{sgn}(v_1^\pm)$, if v_1^\pm is in the same component of $\overline{\mathcal{M}/\mathbb{R}}$ as $w_1 \cup w_2$. On the other hand, if $\dot{\nu}$ points from $w_1 \cup w_2$ towards the interior of $\tilde{\nu}$, then $\widehat{\partial}_s^1 + \widehat{\partial}_s^2$ corresponds to ∂_s^{12} or $-\partial_r$, and $\widehat{\partial}_s^1 - \widehat{\partial}_s^2$ corresponds to $\dot{\nu}$. Hence

$$\begin{aligned} (9.2.7) \quad \mathfrak{o}(D_{w_1\#w_2}) &= -\text{sgn}(w_1) \text{sgn}(w_2) [\widehat{\mathfrak{o}(N)} \wedge \dot{\nu} \wedge \partial_r] \\ &= -\text{sgn}(w_1) \text{sgn}(w_2) [e_1 \wedge \cdots \wedge e_k]. \end{aligned}$$

In particular, $-\text{sgn}(w_1) \text{sgn}(w_2) = \text{sgn}(v_1^\pm)$, if v_1^\pm is in the same component of $\overline{\mathcal{M}/\mathbb{R}}$ as $w_1 \cup w_2$. The signs of $\text{sgn}(w_1) \text{sgn}(w_2)$ in Equations (9.2.6) and (9.2.7) are opposite, as desired.

Finally, we compare the signs of Equations (9.2.6) and (9.2.7) with those of Equation (9.2.5). We contract the right-hand sides of Equations (9.2.6) and (9.2.7) with $e_{k-1}^* \wedge \cdots \wedge e_1^*$ to obtain

$$(-1)^k \text{sgn}(w_1) \text{sgn}(w_2) [-\partial_r] \quad \text{and} \quad (-1)^{k-1} \text{sgn}(w_1) \text{sgn}(w_2) [\partial_r].$$

These agree with

$$(-1)^k \text{sgn}(v_1^+) [-\partial_r] \quad \text{and} \quad (-1)^k \text{sgn}(v_1^-) [\partial_r],$$

which are obtained from Equation (9.2.5) by dividing by $\text{sgn}(v_0)$.

This finishes the proof of Theorem 7.1.1.

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